PARAMETRIZED HOMOLOGY VIA ZIGZAG PERSISTENCE

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Abstract. This paper develops the idea of homology for 1-parameter families of topological spaces. We express parametrized homology as a collection of real intervals with each corresponding to a homological feature supported over that interval or, equivalently, as a persistence diagram. By defining persistence in terms of finite rectangle measures, we classify barcode intervals into four classes. Each of these conveys how the homological features perish at both ends of the interval over which they are defined.

Keywords. persistence diagram, zigzag persistence, levelset zigzag persistence, extended persistence

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1. Introduction

We wish to use the methods of persistent homology [12] to study topological spaces fibered over the real line $\mathbb{R}$. Let X be a topological space and let $f : X \to \mathbb{R}$ be a continuous function. The pair $\mathbb{X} = (X, f)$ is called an $\mathbb{R}$-*space*.

We can view an $\mathbb{R}$-space as a family of topological spaces

$$(\mathbb{X}_a^a = f^{-1}(a) \mid a \in \mathbb{R})$$

called the *levelsets* or *fibers* of $\mathbb{X}$; the topology on the total space X is what gives this collection of spaces the structure of a 'family'. The basic question is to understand the homological invariants of $\mathbb{X}$. In particular, how does the homology of $\mathbb{X}_a^a$ vary with a? Taking the family structure into account, this question demands a richer answer than simply recording the homology of each $\mathbb{X}_a^a$ separately.

What we would like is a reasonable theory that will take an $\mathbb{R}$-space and decompose its homological information into discrete features supported over intervals. In some special cases this has already been done.

First of all, there is the theory of 'extended persistence' introduced by Cohen-Steiner et al. [19] and developed in some detail by Bendich et al. [6] which addresses this matter under the restriction that $\mathbb{X}$ be 'tame' in the sense of having finitely many critical values and cylindrical behavior, ie. 'Morse-like' behaviour, between those critical values. Using *sublevelsets* and *superlevelsets*

$$\mathbb{X}^a = f^{-1}(-\infty, a], \qquad \mathbb{X}_a = f^{-1}[a, +\infty),$$

Cohen-Steiner et al. consider the sequence of spaces and pairs

$$\mathbb{X}^{a_0} \longrightarrow \cdots \longrightarrow \mathbb{X}^{a_n} \longrightarrow X \longrightarrow (X, \mathbb{X}_{a_n}) \longrightarrow \cdots \longrightarrow (X, \mathbb{X}_{a_0})$$

where $a_0 < \cdots < a_n$ is the set of critical values and where the arrows denote the canonical inclusion maps. The extended persistence of $\mathbb{X}$ is defined to be the persistent homology of this sequence; that is to say, the diagram of vector spaces and linear maps

$$\mathrm{H}_1(\mathbb{X}^{a_0}) \longrightarrow \cdots \longrightarrow \mathrm{H}_1(\mathbb{X}^{a_n}) \longrightarrow \mathrm{H}_1(X) \longrightarrow \mathrm{H}_1(X, \mathbb{X}_{a_n}) \longrightarrow \cdots \longrightarrow \mathrm{H}_1(X, \mathbb{X}_{a_0})$$

obtained by applying a homology functor H with field coefficients. The structure of such a diagram is described by its *barcode* or *persistence diagram* (section 2.1). If we fix the homology theory and the field of coefficients, and vary the homological dimension k, then it turns out that the resulting collection of barcodes captures all the information that we are seeking in the present work. The four types of bars identified in [19] each have different geometric significance. We will say more about extended persistence in section 3.9.

Earlier, and in a similar spirit, Dey and Wenger [21] proposed a theory of 'interval persistence'. More recently, Burghelea and Dey [9] initiated the study of persistence invariants of $\mathbb{S}^1$-spaces; that is, of spaces equipped with a continuous function to the circle $\mathbb{S}^1$. This work has been continued and developed by Burghelea and Haller [10].

In this paper we build on the work of Carlsson, de Silva and Morozov [14] and their theory of levelset zigzag persistence. We take advantage of finite rectangle measures to define persistence diagrams [17]. This allows us to work with a broader class of parametrized spaces and prove statements about their diagrams using arguments of a finite nature. Parametrized homology consists of four persistence diagrams that we construct using this method. Each

diagram represents a set of homological features and carries information about how they perish at both ends of the interval over which they are defined. At each end a k-dimensional cycle either expires or is killed in homology by a $(k+1)$-dimensional chain. We prove stability and establish an equivalence between parametrized homology and levelset zigzag persistence, as well as how parametrized homology relates to extended persistence. Using this approach we also define parametrized cohomology.

2. Algebraic Tools

In this section, we review the tools from [13] and [17] that we use to develop parametrized homology invariants.

2.1. **Zigzag modules.** A *zigzag module* $\mathbb{V}$ of length n (see [13]) is a sequence of vector spaces and linear maps between them

$$V_1 \longleftrightarrow V_2 \longleftrightarrow \cdots \longleftrightarrow V_n.$$

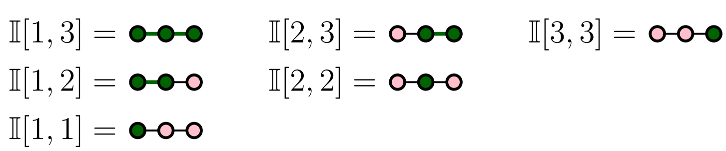
Each $\longleftrightarrow$ represents either a forward map $\longrightarrow$ or a backward map $\longleftarrow$. The particular choice of directions for a given zigzag module is called its *shape*. If every map is a forward map the zigzag module is called a *persistence module* [15].

The basic building blocks of zigzag modules are the interval modules. Fix a shape of length n. The interval module $\mathbb{I}[p,q]$ of that shape is the zigzag module

$$I_1 \longleftrightarrow I_2 \longleftrightarrow \cdots \longleftrightarrow I_n$$

where $I_i = \mathbf{k}$ for $p \leq i \leq q$, and $I_i = 0$ otherwise, and where every $\mathbf{k} \longrightarrow \mathbf{k}$ or $\mathbf{k} \longleftarrow \mathbf{k}$ is the identity map.

Example 2.1. Let $\mathbb{V}_{\{1,2,3\}} = V_1 \longrightarrow V_2 \longrightarrow V_3$. The six interval modules over $\mathbb{V}$ may be represented pictorially as follows:



Each dark green node represents a copy of the field $\mathbf{k}$ and each light pink node represents a copy of the zero vector space. Identity maps are represented by thickened green lines.

A theorem of Gabriel [25] implies that any finite-dimensional zigzag module can be decomposed as a direct sum of interval modules. The extension to infinite-dimensional zigzag modules follows from a theorem of Auslander [3]. The list of summands that appear in the decomposition is an isomorphism invariant of $\mathbb{V}$ by the Krull–Schmidt–Azumaya theorem [4]. We call this isomorphism invariant the *zigzag persistence* of $\mathbb{V}$.

Example 2.2. Consider a zigzag diagram $\mathbb{X}$ of topological spaces and continuous maps between them:

$$X_1 \longleftrightarrow X_2 \longleftrightarrow \cdots \longleftrightarrow X_n$$

We get a zigzag module $\mathrm{H}\mathbb{X}$ by applying a homology functor $\mathrm{H} = \mathrm{H}_j(-;\mathbf{k})$ to this diagram. Decomposing the diagram, we can write

$$\mathrm{H}_j(X_1) \longleftrightarrow \mathrm{H}_j(X_2) \longleftrightarrow \cdots \longleftrightarrow \mathrm{H}_j(X_n) \quad \cong \quad \textstyle\bigoplus_{i\in I} \mathbb{I}[p_i, q_i].$$

The zigzag persistent homology of $\mathbb{X}$ (for the functor H) is then the multiset of intervals $[p_i, q_i]$ in the interval decomposition.

Definition 2.3. The *multiplicity* of an interval $[p, q]$ in a zigzag module $\mathbb{V}$ is the number of copies of $\mathbb{I}[p, q]$ that occur in the interval decomposition of $\mathbb{V}$. This number is written

$$\langle [p, q] \mid \mathbb{V}\rangle$$

and takes values in the set $\{0, 1, 2, \ldots, \infty\}$. (For our purposes we do not need to distinguish different infinite cardinals.) Finally, the *persistence diagram* of $\mathbb{V}$ is the multiset

$$\mathrm{Dgm}(\mathbb{V}) \quad \text{in} \quad \{(p, q) \mid 1 \leq p \leq q \leq n\}$$

defined by the multiplicity function $(p, q) \mapsto \langle [p, q] \mid \mathbb{V}\rangle$.

We will often use pictorial notation for these multiplicities. For example, given a persistence module $\mathbb{V} = \; V_1 \longrightarrow V_2 \longrightarrow V_3$ we may write

$$\langle [2, 3] \mid \mathbb{V}\rangle \quad \text{or} \quad \langle \circ\!-\!\bullet\!-\!\bullet \mid \mathbb{V}\rangle \quad \text{or simply} \quad \langle \circ\!-\!\bullet\!-\!\bullet \rangle$$

for the multiplicity of $\mathbb{I}[2, 3]$ in $\mathbb{V}$.

2.2. **Two calculation principles.** There are two methods from [13] that we repeatedly use to calculate multiplicities: the Restriction Principle and the Diamond Principle.

Theorem 2.4 (Restriction Principle)**.** *Let $\mathbb{V}$ be a zigzag module with two consecutive maps in the same direction*

$$V_1 \longleftrightarrow V_2 \longleftrightarrow \cdots \longleftrightarrow V_{k-1} \xrightarrow{g} V_k \xrightarrow{h} V_{k+1} \leftrightarrow \cdots \longleftrightarrow V_n$$

and let $\mathbb{W}$ be the zigzag module

$$V_1 \longleftrightarrow V_2 \longleftrightarrow \cdots \longleftrightarrow V_{k-1} \xrightarrow{\quad hg \quad} V_{k+1} \leftrightarrow \cdots \longleftrightarrow V_n$$

obtained by combining those maps into a single composite map and deleting the intermediate vector space V_k. Let $[p, q]$ be an interval over the index set for $\mathbb{W}$ (so $p, q \neq k$). Then

$$\langle [p, q] \,|\, \mathbb{W}\rangle = \sum_{[\hat{p},\hat{q}]} \langle [\hat{p}, \hat{q}] \,|\, \mathbb{V}\rangle$$

where the sum is over those intervals $[\hat{p}, \hat{q}]$ over the index set for $\mathbb{V}$ that restrict to $[p, q]$ over the index set of $\mathbb{W}$.

Proof. Take an arbitrary interval decomposition of $\mathbb{V}$. This induces an interval decomposition of $\mathbb{W}$. Summands of $\mathbb{W}$ of type $[p, q]$ arise precisely from summands of $\mathbb{V}$ of types $[\hat{p}, \hat{q}]$ that restrict to $[p, q]$ over the index set of $\mathbb{W}$. $\square$

Example 2.5. Consider a zigzag module

$$\mathbb{V} = \; V_1 \longrightarrow V_2 \longrightarrow V_3 \longleftarrow V_4 \longleftarrow V_5$$

and its restrictions

$$\mathbb{V}_{1,2,3,5} = \; V_1 \longrightarrow V_2 \longrightarrow V_3 \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!- V_5$$

$$\mathbb{V}_{1,3,4,5} = \; V_1 -\!\!\!-\!\!\!-\!\!\!-\!\!\!\longrightarrow V_3 \longleftarrow V_4 \longleftarrow V_5$$

obtained in the manner described above. Then

$$\langle \circ\!-\!\circ\!-\!\bullet\!-\!\!-\!\!-\!\bullet \mid \mathbb{V}_{1,2,3,5} \rangle = \langle \circ\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet \mid \mathbb{V} \rangle,$$
$$\langle \circ\!-\!\!-\!\!-\!\bullet\!-\!\bullet\!-\!\bullet \mid \mathbb{V}_{1,3,4,5} \rangle = \langle \circ\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet \mid \mathbb{V} \rangle + \langle \circ\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet \mid \mathbb{V} \rangle.$$

The extra term occurs when the interval for the restricted module abuts the long edge on either side (so there is both a clear node and a filled node at that edge). There are then two possible intervals which restrict to it.

The Diamond Principle relates the interval multiplicities of zigzag modules that are related by a different kind of local change. The principle is most sharply expressed in terms of the reflection functors of Bernstein, Gelfand and Ponomarev [7]. We make do with a simpler non-functorial statement. We say that a diamond-shaped commuting diagram of vector spaces

$$\begin{tikzcd} & D & \\ B \arrow[ur, "j_1"] & & C \arrow[ul, "j_2"'] \\ & A \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & \end{tikzcd}$$

is *exact* if the sequence

$$A \xrightarrow{i_1 \oplus i_2} B \oplus C \xrightarrow{j_1 - j_2} D$$

is exact at $B \oplus C$. This means that a pair of vectors $\beta \in B$, $\gamma \in C$ satisfies $j_1(\beta) = j_2(\gamma)$ if and only if there exists $\alpha \in A$ such that $\beta = i_1(\alpha)$ and $\gamma = i_2(\alpha)$.

Theorem 2.6 (Diamond Principle [13])**.** *Consider a diagram of vector spaces*

$$\begin{tikzcd} & & & V_k^+ & & & \\ V_1 \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & V_{k-2} \arrow[r, leftrightarrow] & V_{k-1} \arrow[u] & V_{k+1} \arrow[ul] \arrow[r, leftrightarrow] & V_{k+2} \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & V_n \\ & & & V_k^- \arrow[u] \arrow[ur] & & & \end{tikzcd}$$

where the middle diamond is exact. Let $\mathbb{V}^+, \mathbb{V}^-$ respectively denote the upper zigzag module (containing V_k^+) and the lower zigzag module (containing V_k^-) in this diagram. Then the following multiplicities are equal.

(i) If the interval $[p, q]$ does not meet $\{k-1, k, k+1\}$ then

$$\langle [p,q] \mid \mathbb{V}^+ \rangle = \langle [p,q] \mid \mathbb{V}^- \rangle.$$

(ii) If the interval $[p,q]$ completely contains $\{k-1,k,k+1\}$ then

$$\langle [p,q] \mid \mathbb{V}^+\rangle = \langle [p,q] \mid \mathbb{V}^-\rangle.$$

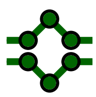
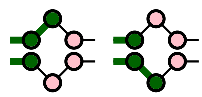
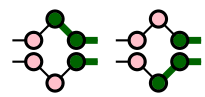
(iii) For $p \leq k-1$ we have

$$\begin{aligned}\langle [p,k] \quad\ \ \mid \mathbb{V}^+\rangle &= \langle [p,k-1] \mid \mathbb{V}^-\rangle,\\ \langle [p,k-1] \mid \mathbb{V}^+\rangle &= \langle [p,k] \quad\ \ \mid \mathbb{V}^-\rangle.\end{aligned}$$

(iv) For $q \geq k+1$ we have

$$\begin{aligned}\langle [k,q] \quad\ \ \mid \mathbb{V}^+\rangle &= \langle [k+1,q] \mid \mathbb{V}^-\rangle,\\ \langle [k+1,q] \mid \mathbb{V}^+\rangle &= \langle [k,q] \quad\ \ \mid \mathbb{V}^-\rangle.\end{aligned}$$

The diagrams

(ii)  *(iii)*  *(iv)* 

express the last three of these rules pictorially. □

Remark. The theorem gives no information about $\langle [k,k] \mid \mathbb{V}^+\rangle$ or $\langle [k,k] \mid \mathbb{V}^-\rangle$. These quantities are independent of each other and of all other multiplicities.

We use the Diamond Principle frequently in the following situation. Consider a diagram of topological spaces of the following form:

$$\begin{tikzcd}[column sep=small] & & & & A \cup B & & & & \\ X_1 \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & X_{k-2} \arrow[r, leftrightarrow] & A \arrow[ur] & & B \arrow[ul] \arrow[r, leftrightarrow] & X_{k+2} \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & X_n \\ & & & & A \cap B \arrow[ul] \arrow[ur] & & & & \end{tikzcd}$$

Here A, B are subspaces of some common ambient space. Applying a homology functor H, we obtain an upper zigzag diagram $\mathbb{V}^\cup$ and a lower zigzag diagram $\mathbb{V}^\cap$. The exactness of the diamond is precisely the exactness of the central term in the following excerpt from the Mayer–Vietoris sequence:

$$\cdots \longrightarrow \mathrm{H}(A\cap B) \longrightarrow \mathrm{H}(A) \oplus \mathrm{H}(B) \longrightarrow \mathrm{H}(A \cup B) \longrightarrow \cdots$$

In situations where the Mayer–Vietoris theorem holds, we can use the Diamond Principle to compare the interval summands of $\mathbb{V}^\cup$ and $\mathbb{V}^\cap$. The reader is reminded that the Mayer–Vietoris theorem is not always applicable. We treat this matter carefully in Section 3.2.

2.3. **Persistence diagrams and measures.** As we discussed in Section 2.1, a zigzag module with a finite index set decomposes essentially uniquely into interval modules. There are finitely many interval module types, so the structure of the zigzag module is determined by a finite list of multiplicities.

On other other hand, the objects we are studying are spaces parametrized over the real line; and so we will want to define continuous-parameter persistence diagrams. The motivating heuristic is that each topological feature will be supported over some interval of $\mathbb{R}$. These intervals may be open, closed or half-open, so we follow Chazal et al. [17] in describing their

endpoints as real numbers *decorated* with a $^+$ or $^-$ superscript. The superscript * may be used for an unspecified decoration. Here are the four options:

interval	decorated pair	point with tick
(p, q)	(p^+, q^-)	
$(p, q]$	(p^+, q^+)	
$[p, q)$	(p^-, q^-)	
$[p, q]$	(p^-, q^+)	

Except for the degenerate interval $[p, p] = (p^-, p^+)$, we require $p < q$. For infinite intervals, we allow $p = -\infty$ and $q = +\infty$ and their decorated forms $p^* = -\infty^+$ and $q^* = +\infty^-$.

Given a collection (i.e. multiset) of such intervals, we can form a persistence diagram by drawing each (p^*, q^*) as a point in the plane with a tick to indicate the decorations. The tick convention is self-explanatory. The diagram resides in the extended half-plane

$$\mathcal{H} = \{(p, q) \mid -\infty \leq p < q \leq \infty\}$$

which we can draw schematically as a triangle. If we omit the ticks (i.e. forget the decorations), what remains is an *undecorated* persistence diagram.

Our main mechanism for defining and studying continuous-parameter persistence modules is taken from [17]: a finite measure theory designed for this task. Define

$$\operatorname{Rect}(\mathcal{H}) = \{[a, b] \times [c, d] \subset \mathcal{H} \mid -\infty \leq a < b < c < d \leq +\infty\}.$$

This consists of finite rectangles, horizontal semi-infinite strips, vertical semi-infinite strips and infinite quadrants in $\mathcal{H}$. A rectangle measure or *r-measure* on $\mathcal{H}$ is a function

$$\mu\colon \operatorname{Rect}(\mathcal{H}) \longrightarrow \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$$

that is additive with respect to splitting a rectangle horizontally or vertically into two rectangles. Explicitly, we require

$$\mu([a, b] \times [c, d]) = \mu([a, p] \times [c, d]) + \mu([p, b] \times [c, d]) \qquad \text{(horizontal split)}$$
$$\mu([a, b] \times [c, d]) = \mu([a, b] \times [c, q]) + \mu([a, b] \times [q, d]) \qquad \text{(vertical split)}$$

whenever $a < p < b < c < q < d$ (see Figure 2.3). By iterating these formulas, it follows that μ must be additive with respect to arbitrary tilings of a rectangle by other rectangles. This implies, in particular, that μ is monotone with respect to inclusion of rectangles.

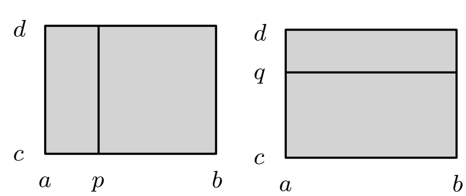


FIGURE 1. Rectangles split horizontally and vertically.

The 'atoms' for this measure theory are decorated points rather than points; when a rectangle is split in two, points along the split line have to be assigned to one side or the

other and this is done using the tick. We write $(p^*, q^*) \in R$ to mean that (p, q) lies in R with the tick pointing into the interior of R (this is automatic for interior points).

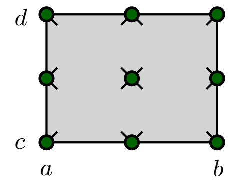


FIGURE 2. A decorated point (p^*, q^*) is contained in R if and only if (p, q) is contained in R and the tick points into the interior.

Theorem 2.7 (Equivalence Theorem [17])**.** *There is a bijective correspondence between*

- *Finite r-measures μ on $\mathcal{H}$; and*
- *Locally finite multisets A of decorated points in $\mathcal{H}$.*

Here 'finite' means that $\mu(R) < \infty$ for all R, and 'locally finite' means that $\mathrm{card}(A|_R) < \infty$ *for all R. Explicitly, a multiset A corresponds to the measure μ defined by the formula*

$$\mu(R) = \mathrm{card}(A|_R),$$

(the cardinality of the multiset of decorated points of A that belong to R); and, conversely, a measure μ corresponds to the multiset A with multiplicity function

$$\mathrm{m}_A(p^*, q^*) = \min\{\mu(R) \mid R \in \mathrm{Rect}(\mathcal{H}) \text{ such that } (p^*, q^*) \in R\}.$$

In other words, finite r-measures correspond exactly to decorated persistence diagrams. □

Remark. Since r-measures are monotone, the 'min' in the formula for m_A can be calculated as a limit. For example

$$\mathrm{m}_A(p^+, q^-) = \lim_{\epsilon \to 0} \mu([p, p + \epsilon] \times [q - \epsilon, q]),$$

with similar formulas for the other choices of decoration for (p^*, q^*) and for points at infinity. Since the expression inside the 'lim' takes values in the natural numbers and decreases as ϵ decreases, it necessarily stabilizes for sufficiently small ϵ.

The multiset A corresponding to a finite r-measure μ is its *decorated diagram*, written $\mathrm{Dgm}(\mu)$. We obtain the *undecorated diagram* $\mathrm{Dgm_u}(\mu)$ by forgetting the decorations. This is a multiset in $\mathcal{H}$.

When the r-measure is not finite, the *finite support* is defined in [17] to be the set of decorated points in $\mathcal{H}$ that are contained in some rectangle of finite measure. Within the finite support there is a well-defined decorated persistence diagram which characterizes the r-measure as above, with the proviso that rectangles which extend beyond the finite support have infinite measure. In particular, the undecorated diagram can be thought of as a locally finite multiset defined in some open set $\mathcal{F} \subseteq \mathcal{H}$ and deemed to have infinite multiplicity everywhere else in the extended plane.

3. Parametrized Homology

In this section we define 'parametrized homology' invariants for $\mathbb{R}$-spaces. Given an $\mathbb{R}$-space $\mathbb{X} = (\mathbb{X}, f)$ and a homology functor H with field coefficients, we define four persistence diagrams

$$\mathrm{Dgm}^{\wedge}(\mathrm{H}\mathbb{X}),\ \mathrm{Dgm}^{\backslash\backslash}(\mathrm{H}\mathbb{X}),\ \mathrm{Dgm}^{//}(\mathrm{H}\mathbb{X}),\ \mathrm{Dgm}^{\vee}(\mathrm{H}\mathbb{X})$$

that detect topological features exhibiting four different behaviors. We will need to impose conditions on H and $\mathbb{X}$ to guarantee that the r-measures used to define these diagrams are additive and finite.

3.1. **Four measures.** Let $\mathbb{X} = (X, f)$ be a parametrized and let H be a homology functor with field coefficients. Given a rectangle

$$R = [a, b] \times [c, d], \qquad -\infty \leq a < b < c < d \leq +\infty,$$

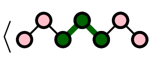
we wish to count the homological features of $\mathbb{X}$ that are supported over the closed interval $[b, c]$ but do not reach either end of the open interval (a, d). Accordingly, consider the diagram

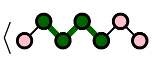
$$\mathbb{X}_{\{a,b,c,d\}} : \begin{tikzcd}[column sep=small] & \mathbb{X}_a^b & & \mathbb{X}_b^c & & \mathbb{X}_c^d & \\ \mathbb{X}_a^a \arrow[ur] & & \mathbb{X}_b^b \arrow[ul] \arrow[ur] & & \mathbb{X}_c^c \arrow[ul] \arrow[ur] & & \mathbb{X}_d^d \arrow[ul] \end{tikzcd}$$

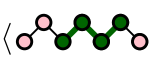
of spaces and inclusion maps, where $\mathbb{X}_a^b = f^{-1}[a, b]$. We assume $\mathbb{X}_{-\infty}^{-\infty}$ and $\mathbb{X}_{+\infty}^{+\infty}$ to be empty if they occur. Apply H to obtain a diagram

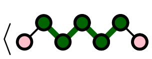
$$\mathrm{H}\mathbb{X}_{\{a,b,c,d\}} : \begin{tikzcd}[column sep=small] & \mathrm{H}(\mathbb{X}_a^b) & & \mathrm{H}(\mathbb{X}_b^c) & & \mathrm{H}(\mathbb{X}_c^d) & \\ \mathrm{H}(\mathbb{X}_a^a) \arrow[ur] & & \mathrm{H}(\mathbb{X}_b^b) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_c^c) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_d^d) \arrow[ul] \end{tikzcd}$$

of vector spaces and linear maps. Decomposing this zigzag module into interval modules, four of the multiplicities are of interest to us. Define four quantities as follows:

$\mu^{\wedge}_{\mathrm{H}\mathbb{X}}(R) = \langle$  $\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}} \rangle$

$\mu^{\backslash\backslash}_{\mathrm{H}\mathbb{X}}(R) = \langle$  $\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}} \rangle$

$\mu^{//}_{\mathrm{H}\mathbb{X}}(R) = \langle$  $\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}} \rangle$

$\mu^{\vee}_{\mathrm{H}\mathbb{X}}(R) = \langle$  $\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}} \rangle$.

Each of these counts topological features of a certain type, supported over $[b, c]$ but not outside (a, d). Under favorable circumstances, these four functions of R turn out to be finite r-measures and therefore their behavior can be completely described by a decorated persistence diagram in the extended half-space. We will identify such circumstances in later parts of this chapter.

The distinction between the four behaviors is seen in Figure 3. Consider 0-dimensional singular homology $\mathrm{H} = \mathrm{H}_0(-; \mathbf{k})$. In each example $\mathrm{H}\mathbb{X}_b^b \cong \mathrm{H}\mathbb{X}_b^c \cong \mathrm{H}\mathbb{X}_c^c$ have rank two whereas $\mathrm{H}\mathbb{X}_a^a$, $\mathrm{H}\mathbb{X}_d^d$ each have rank one. The way in which the second feature (i.e. the second connected component) perishes at each end is determined by the ranks of the maps

$$\mathrm{H}\mathbb{X}_a^b \longleftarrow \mathrm{H}\mathbb{X}_b^b \quad \text{and} \quad \mathrm{H}\mathbb{X}_c^c \longrightarrow \mathrm{H}\mathbb{X}_c^d\ .$$

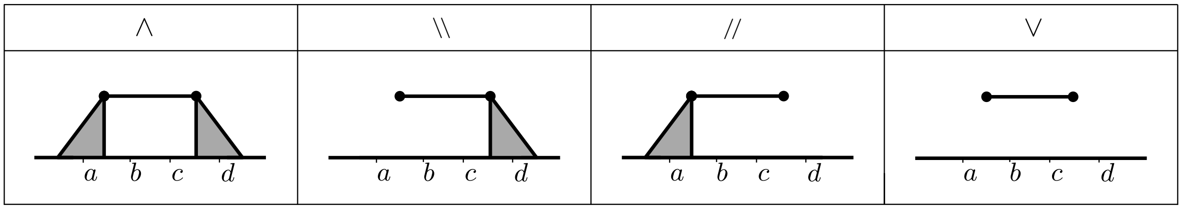
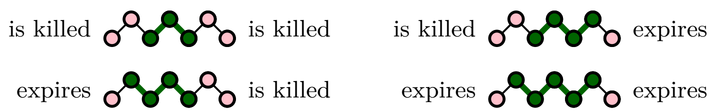


FIGURE 3. Two components (over $[b, c]$) become one (over a and d). The four ways this can happen are detected by $\mu^{\wedge}, \mu^{\backslash\backslash}, \mu^{/\!/}, \mu^{\vee}$ respectively.

If the rank is two, then the feature has simply *expired* at that end: it is no longer there at $\mathbb{X}_a^a$ or $\mathbb{X}_d^d$. If the rank is one, that means the feature has been *killed* by some 1-cell that has appeared in $\mathbb{X}_a^b$ or $\mathbb{X}_c^d$. In terms of zigzag summands, the situation looks like this:



Our definitions associate the four symbols $\wedge$, $\backslash\backslash$, $/\!/$, $\vee$ with these four behaviors. An unspecified behavior may be indicated by the symbol ⩙.

Proposition 3.1. *The four behaviors have 'coordinate-reversal' symmetry. Specifically, suppose* $\mathbb{X} = (X, f)$ *and* $R = [a, b] \times [c, d]$. *If we define the coordinate reversals* $\overline{\mathbb{X}} = (X, -f)$ *and* $\overline{R} = [-d, -c] \times [-b, -a]$ *then the relations*

$$\mu^{\wedge}_{\mathrm{H}\overline{\mathbb{X}}}(\overline{R}) = \mu^{\wedge}_{\mathrm{H}\mathbb{X}}(R) \qquad \mu^{/\!/}_{\mathrm{H}\overline{\mathbb{X}}}(\overline{R}) = \mu^{\backslash\backslash}_{\mathrm{H}\mathbb{X}}(R)$$
$$\mu^{\backslash\backslash}_{\mathrm{H}\overline{\mathbb{X}}}(\overline{R}) = \mu^{/\!/}_{\mathrm{H}\mathbb{X}}(R) \qquad \mu^{\vee}_{\mathrm{H}\overline{\mathbb{X}}}(\overline{R}) = \mu^{\vee}_{\mathrm{H}\mathbb{X}}(R)$$

follow immediately. □

Our next step is to identify when the four functions $\mu^{\text{⩙}}_{\mathrm{H}\mathbb{X}}(R)$ are finite r-measures. We consider additivity first (Sections 3.2 and 3.3), then finiteness (Section 3.4).

3.2. **Tautness.** In proving additivity and other identities, we will make much use of the Diamond Principle. For $p < q < r < s$, consider the following diamonds:

$$\begin{tikzcd}[column sep=small] & \mathrm{H}_k(\mathbb{X}_p^s) & \\ \mathrm{H}_k(\mathbb{X}_p^r) \arrow[ur] & & \mathrm{H}_k(\mathbb{X}_q^s) \arrow[ul] \\ & \mathrm{H}_k(\mathbb{X}_q^r) \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & \mathrm{H}_k(\mathbb{X}_p^r) & \\ \mathrm{H}_k(\mathbb{X}_p^q) \arrow[ur] & & \mathrm{H}_k(\mathbb{X}_q^r) \arrow[ul] \\ & \mathrm{H}_k(\mathbb{X}_q^q) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The exactness of the left diamond is guaranteed by the Mayer–Vietoris theorem, which applies because the relative interiors of $\mathbb{X}_p^r, \mathbb{X}_q^s$ contain the sets $f^{-1}[p, r), f^{-1}(q, s]$ which cover $\mathbb{X}_p^s$. In contrast, there is no such guarantee for the right diamond: the relative interiors of $\mathbb{X}_p^q, \mathbb{X}_q^r$ do not cover $\mathbb{X}_p^r$.

We identify a local condition on the embedding of $\mathbb{X}_q^q$ in $\mathbb{X}$, in terms of the homology theory H, which gives us exactness of all such diamonds. Let U be any neighborhood of $\mathbb{X}_q^q$

(such as $\mathbb{X}_p^r$). It splits into two parts: a lower-neighborhood

$$A = U \cap \mathbb{X}^q = U \cap f^{-1}(-\infty, q],$$

and an upper-neighborhood

$$B = U \cap \mathbb{X}_q = U \cap f^{-1}[q, +\infty).$$

Then $U = A \cup B$ and $\mathbb{X}_q^q = A \cap B$, and we desire the exactness of

$$(\diamond_{AB}) \qquad \begin{tikzcd} & \mathrm{H}_k(U) & \\ \mathrm{H}_k(A) \arrow[ur] & & \mathrm{H}_k(B) \arrow[ul] \\ & \mathrm{H}_k(\mathbb{X}_q^q) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in whichever dimension k we are considering. Here are two criteria.

Criterion A. *The levelset $\mathbb{X}_q^q$ is* H_k-taut in U *if the map (induced by inclusion)*

$$\alpha_{k+1} \colon \mathrm{H}_{k+1}(A, \mathbb{X}_q^q) \to \mathrm{H}_{k+1}(U, B)$$

is an epimorphism, and the map (induced by inclusion)

$$\alpha_k \colon \mathrm{H}_k(A, \mathbb{X}_q^q) \to \mathrm{H}_k(U, B)$$

is a monomorphism.

Criterion B. *The levelset $\mathbb{X}_q^q$ is* H_k-taut in U *if the map (induced by inclusion)*

$$\beta_{k+1} \colon \mathrm{H}_{k+1}(B, \mathbb{X}_q^q) \to \mathrm{H}_{k+1}(U, A)$$

is an epimorphism, and the map (induced by inclusion)

$$\beta_k \colon \mathrm{H}_k(B, \mathbb{X}_q^q) \to \mathrm{H}_k(U, A)$$

is a monomorphism.

The maps α_*, β_* are excision maps, and they would automatically be isomorphisms if the excision axiom applied to them. For the axiom to apply we would need

$$\mathrm{closure}(B - \mathbb{X}_q^q) \subseteq \mathrm{interior}(B)$$
$$\mathrm{closure}(A - \mathbb{X}_q^q) \subseteq \mathrm{interior}(A)$$

for α_*, β_* respectively, and this is not true in general.

Proposition 3.2. *The two criteria are equivalent.*

Proof. We show that the statements for α_{k+1}, α_k together imply the statements for β_{k+1}, β_k (the converse being symmetric).

The following commutative diagram is obtained by criss-crossing the long exact sequences for the triples $(U, A, \mathbb{X}_q^q)$ and $(U, B, \mathbb{X}_q^q)$:

$$\begin{tikzcd}[column sep=tiny]
\mathrm{H}_{k+1}(A, \mathbb{X}_q^q) \arrow[rr, "\alpha_{k+1}"] \arrow[dr] & & \mathrm{H}_{k+1}(U, B) \arrow[rr, "\partial"] & & \mathrm{H}_k(B, \mathbb{X}_q^q) \arrow[rr, "\beta_k"] \arrow[dr] & & \mathrm{H}_k(U, A) \\
& \mathrm{H}_{k+1}(U, \mathbb{X}_q^q) \arrow[ur] \arrow[dr] & & & & \mathrm{H}_k(U, \mathbb{X}_q^q) \arrow[ur] \arrow[dr] & \\
\mathrm{H}_{k+1}(B, \mathbb{X}_q^q) \arrow[rr, "\beta_{k+1}"'] \arrow[ur] & & \mathrm{H}_{k+1}(U, A) \arrow[rr, "\partial"'] & & \mathrm{H}_k(A, \mathbb{X}_q^q) \arrow[rr, "\alpha_k"'] \arrow[ur] & & \mathrm{H}_k(U, B)
\end{tikzcd}$$

Note that α_{k+1} being an epimorphism implies that the upper ∂ is zero, and α_k being a monomorphism implies that the lower ∂ is zero. With that in mind, it becomes a routine diagram-chase to show that β_{k+1} is an epimorphism and β_k is a monomorphism. $\square$

We use the term *normal neighborhood* to refer to a neighborhood which contains a closed neighborhood. In a normal topological space (such as a compact Hausdorff space), all neighborhoods of a closed set are normal. Closed neighborhoods are trivially normal.

Proposition 3.3. *If the levelset* $\mathbb{X}_q^q$ *is* H_k*-taut in some normal neighborhood, then it is* H_k*-taut in any normal neighborhood.*

Proof. Since any two normal neighborhoods contain a closed neighborhood in common, it is enough to show that

$$\mathbb{X}_q^q \text{ is } \mathrm{H}_k\text{-taut in } U \quad \Leftrightarrow \quad \mathbb{X}_q^q \text{ is } \mathrm{H}_k\text{-taut in } W$$

whenever $U \subseteq W$ are neighborhoods and U is closed. Writing $U = A \cup B$ and $W = A' \cup B'$ as usual, we also consider $V = A \cup B'$.

Criterion A gives the same result for U as for V, by considering

$$\mathrm{H}_*(A, \mathbb{X}_q^q) \to \mathrm{H}_*(A \cup B, A) \xrightarrow{\simeq} \mathrm{H}_k(A \cup B', B')$$

The right-hand map is an isomorphism by the excision axiom, which applies in this situation because $A \cup B$ is a closed neighborhood of A in $A \cup B'$.

Criterion B gives the same result for V as for W, by considering

$$\mathrm{H}_*(B', \mathbb{X}_q^q) \to \mathrm{H}_*(A \cup B', A) \xrightarrow{\simeq} \mathrm{H}_k(A' \cup B', A')$$

The right-hand map is an isomorphism by excision, since $A \cup B'$ is a closed neighborhood of B' in $A' \cup B'$.

The result follows. $\square$

Definition 3.4. Accordingly, we say that the levelset $\mathbb{X}_q^q$ is H_k*-taut* if it is H_k-taut in some, and therefore every, normal neighborhood.

Definition 3.5. We say that the levelset $\mathbb{X}_q^q$ is H*-taut* if it is H_k-taut in all dimensions k. This means that for every normal neighborhood U, the maps

$$\alpha_k : \mathrm{H}_k(A, \mathbb{X}_q^q) \to \mathrm{H}_k(U, B)$$

are isomorphisms for all k, or equivalently

$$\beta_k : \mathrm{H}_k(B, \mathbb{X}_q^q) \to \mathrm{H}_k(U, A)$$

are isomorphisms for all k.

Proposition 3.6. *If the levelset $\mathbb{X}_q^q$ is H_k-taut, then the diagram $(\diamond_{AB})$ is exact for any normal neighborhood $U = A \cup B$.*

Proof. Using Criterion B, say, this is a straightforward chase on the diagram

$$\begin{tikzcd}
\mathrm{H}_{k+1}(U, A) \arrow[r] & \mathrm{H}_k(A) \arrow[r] & \mathrm{H}_k(U) \arrow[r] & \mathrm{H}_k(U, A) \\
\mathrm{H}_{k+1}(B, \mathbb{X}_q^q) \arrow[r] \arrow[u, "\text{epi}"] & \mathrm{H}_k(\mathbb{X}_q^q) \arrow[r] \arrow[u] & \mathrm{H}_k(B) \arrow[r] \arrow[u] & \mathrm{H}_k(B, \mathbb{X}_q^q) \arrow[u, "\text{mono}"']
\end{tikzcd}$$

for the map of long exact sequences induced by the inclusion $(B, \mathbb{X}_q^q) \to (U, A)$. □

This completes our treatment of tautness. Here are some examples.

Proposition 3.7. *The $\mathbb{R}$-space $\mathbb{X} = (X, f)$ has* H*-taut levelsets under any of the following circumstances:*

(i) X is locally compact, f is proper, and H *is Steenrod–Sitnikov homology* [24, 28].

(ii) Each $\mathbb{X}_q^q$ is a deformation retract of some closed neighborhood in $\mathbb{X}_q$ or $\mathbb{X}^q$.

(iii) X is a smooth manifold and f is a proper Morse function.

(iv) X is a locally compact polyhedron and f is a proper piecewise-linear map.

(v) $X \subseteq \mathbb{R}^n \times \mathbb{R}$ is a closed definable set in some o-minimal structure [29] *and f is the projection onto the second factor. In particular, this applies when X is semialgebraic* [5].

Proof. *(i)* Steenrod–Sitnikov homology satisfies a strengthened form of excision axiom [28] that does not require any restraints on sets. Therefore maps in Definition 3.5 are isomorphisms for any levelset $\mathbb{X}_q^q$.

(ii) Let C_1 be a closed neighborhood of $\mathbb{X}_q^q$. We know $\mathbb{X}_q^q$ is a deformation retract of a closed neighborhood C_2 in $\mathbb{X}^q$. We may assume without loss of generality that $C_2 \subseteq C_1$. Let $C = C_2 \cup (C_1 \cap \mathbb{X}_q)$. The homology groups $\mathrm{H}_k(C_2, \mathbb{X}_q^q)$ and $\mathrm{H}_k(C, C \cap \mathbb{X}_q)$ are trivial for every k and therefore isomorphic, implying that $\mathbb{X}_q^q$ is H-taut.

(iii), *(iv)* and *(v)* follow from *(ii)*. In particular, we prove *(v)* by applying [29, Corollary 3.9, Chapter 8]. □

Remark. We occasionally need to consider Mayer–Vietoris diamonds in relative homology. We establish their exactness individually as they occur.

3.3. **Additivity.** We are now ready to prove that the four measures $\mu_{\mathrm{H}\mathbb{X}}^{\between}$ are additive.

Theorem 3.8. *Let* H *be a homology functor with field coefficients and let $\mathbb{X} = (X, f)$ be an $\mathbb{R}$-space whose levelsets are* H*-taut. Then $\mu_{\mathrm{H}\mathbb{X}}^{\wedge}$, $\mu_{\mathrm{H}\mathbb{X}}^{\backslash\backslash}$, $\mu_{\mathrm{H}\mathbb{X}}^{//}$, and $\mu_{\mathrm{H}\mathbb{X}}^{\vee}$ are additive.*

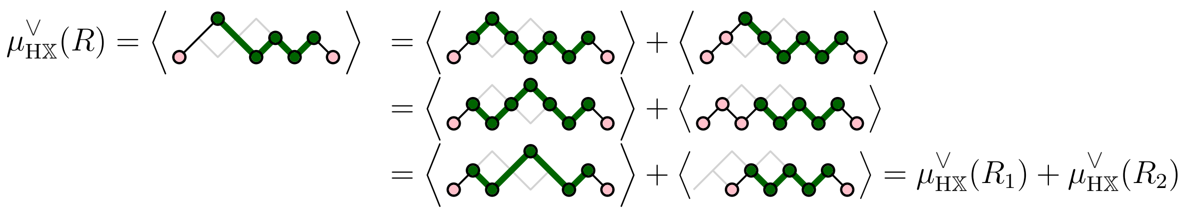
Proof. Let $R = [a, b] \times [c, d]$ and consider a horizontal split

$$R_1 = [a, p] \times [c, d], \qquad R_2 = [p, b] \times [c, d],$$

so $a < p < b < c < d$. The diagram

$$\begin{tikzcd}[column sep=small]
 & & \mathbb{X}_a^b & & \mathbb{X}_p^c & & & & \\
 & \mathbb{X}_a^p \arrow[ur] & & \mathbb{X}_p^b \arrow[ul] \arrow[ur] & & \mathbb{X}_b^c \arrow[ul] & & \mathbb{X}_c^d & \\
\mathbb{X}_a^a \arrow[ur] & & \mathbb{X}_p^p \arrow[ul] \arrow[ur] & & \mathbb{X}_b^b \arrow[ul] \arrow[ur] & & \mathbb{X}_c^c \arrow[ul] \arrow[ur] & & \mathbb{X}_d^d \arrow[ul]
\end{tikzcd}$$

contains the zigzags $\mathbb{X}_{\{a,b,c,d\}}$, $\mathbb{X}_{\{a,p,c,d\}}$, $\mathbb{X}_{\{p,b,c,d\}}$ for all three rectangles. When we apply H, the two diamonds in the resulting diagram are exact since the levelsets $\mathbb{X}_p^p, \mathbb{X}_b^b$ are H-taut. We calculate:



In the first line we add two extra nodes to refine the 7-term zigzag to a 9-term zigzag and use the Restriction Principle. In the second line we use the Diamond Principle twice. In the third line we drop two nodes in each term and use the Restriction Principle again.

Similar calculations establish the additivity of $\mu^{\wedge}_{\mathrm{H}\mathbb{X}}$, $\mu^{\backslash\backslash}_{\mathrm{H}\mathbb{X}}$ and $\mu^{//}_{\mathrm{H}\mathbb{X}}$ under horizontal splitting. Additivity under vertical splitting follows by coordinate-reversal symmetry. $\square$

3.4. **Finiteness.** We now consider the finiteness of the four r-measures $\mu^{\times}_{\mathrm{H}\mathbb{X}}$. As discussed in Section 2.3, finiteness of an r-measure implies that its decorated persistence diagram is defined everywhere in $\mathcal{H}$; in general the diagram is defined in the finite support of the r-measure.

It turns out to be essentially the same issue as the finiteness of the *well groups* [6, 22]. Well groups measure that part of the homology of a fiber $\mathrm{H}(\mathbb{X}_m^m)$ of an $\mathbb{R}$-space that is stable under ϵ-perturbations of the coordinate. One defines

$$\mathrm{W}(\mathrm{H}\mathbb{X}; m, \epsilon) = \bigcap_g \mathrm{image}\left[\mathrm{H}(g^{-1}(q)) \longrightarrow \mathrm{H}\mathbb{X}_{q-\epsilon}^{q+\epsilon}\right]$$

where the intersection is taken over all ϵ-perturbations g of the coordinate f, perhaps in a suitable regularity class. Considering the perturbations $g = f \pm \epsilon$, it follows that the well group is contained in[1]

$$\mathrm{image}\left[\mathrm{H}\mathbb{X}_{q-\epsilon}^{q-\epsilon} \longrightarrow \mathrm{H}\mathbb{X}_{q-\epsilon}^{q+\epsilon}\right] \cap \mathrm{image}\left[\mathrm{H}\mathbb{X}_{q+\epsilon}^{q+\epsilon} \longrightarrow \mathrm{H}\mathbb{X}_{q-\epsilon}^{q+\epsilon}\right]$$

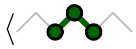
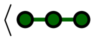
and therefore its rank is bounded by

$$\langle \bullet\!-\!\bullet\!-\!\bullet \mid \mathrm{H}_{q-\epsilon}^{q-\epsilon} \longrightarrow \mathrm{H}_{q-\epsilon}^{q+\epsilon} \longleftarrow \mathrm{H}_{q+\epsilon}^{q+\epsilon} \rangle = \langle \bullet\!-\!\bullet\!-\!\bullet \mid \mathrm{H}\mathbb{X}_{\{q-\epsilon, q+\epsilon\}} \rangle.$$

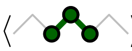
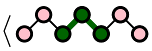
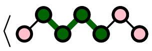
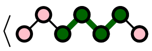
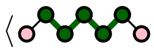
This takes the same form as the term that we need to bound.

[1]Indeed, the well group is equal to this intersection if the class of perturbations has H-taut fibers.

Lemma 3.9. *Let $\mathbb{X} = (X, f)$ be an $\mathbb{R}$-space and H be a homology functor. For any rectangle $R = [a, b] \times [c, d]$ with $a < b < c < d$ we have*

$$\mu^{\wedge}_{\mathrm{H}\mathbb{X}}(R) + \mu^{\backslash\backslash}_{\mathrm{H}\mathbb{X}}(R) + \mu^{//}_{\mathrm{H}\mathbb{X}}(R) + \mu^{\vee}_{\mathrm{H}\mathbb{X}}(R) \leq \langle$$  $$\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}}\rangle = \langle$$  $$\mid \mathrm{H}\mathbb{X}_{\{b,c\}}\rangle.$$

Proof. By the Restriction Principle

$$\langle$$  $$\rangle \geq \langle$$  $$\rangle + \langle$$  $$\rangle + \langle$$  $$\rangle + \langle$$  $$\rangle$$

$$= (\mu^{\wedge}_{\mathrm{H}\mathbb{X}} + \mu^{\backslash\backslash}_{\mathrm{H}\mathbb{X}} + \mu^{//}_{\mathrm{H}\mathbb{X}} + \mu^{\vee}_{\mathrm{H}\mathbb{X}})(R).$$ □

Proposition 3.10. *Let $\mathbb{X} = (X, f)$. Then $\mu^{\vee}_{\mathrm{H}\mathbb{X}}$, $\mu^{//}_{\mathrm{H}\mathbb{X}}$, $\mu^{\wedge}_{\mathrm{H}\mathbb{X}}$ and $\mu^{\backslash\backslash}_{\mathrm{H}\mathbb{X}}$ are finite for any H under any of the following circumstances:*

(i) X is a locally compact polyhedron and f a proper continuous map.

(iii) X is a smooth manifold and f is a proper Morse function.

(iv) X is a locally compact polyhedron and f is a proper piecewise-linear map.

(v) $X \subseteq \mathbb{R}^n \times \mathbb{R}$ is a closed definable set in some o-minimal structure and f is the projection onto the second factor.

Proof. In cases (iii), (iv), (iv) each slice $\mathbb{X}^c_b$ has the homotopy type of a finite cell complex, and therefore has finite-dimensional homology.

The proof of (i) is a little more involved. Let $R = [a, b] \times [c, d]$. Choose m and $\epsilon > 0$ such that $b + 2\epsilon < m < c - 2\epsilon$, and approximate f with a piecewise-linear map $g \colon X \to \mathbb{R}$ for which $\|g - f\| \leq \epsilon$. Then g is also proper, and $Y = g^{-1}(m)$ is triangulable as a finite simplicial complex and is H-taut as a fiber of (X, g).

We can split the neighborhood $\mathbb{X}^c_b$ into lower- and upper-neighborhoods of Y by defining

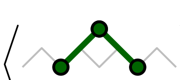
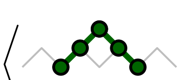
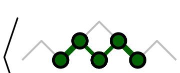
$$U = \mathbb{X}^c_b \cap g^{-1}(-\infty, m], \quad V = \mathbb{X}^c_b \cap g^{-1}[m, +\infty).$$

Thus $\mathbb{X}^c_b = U \cup V$ and $Y = U \cap V$. Since $||g - f|| \leq \epsilon$, we also have $\mathbb{X}^b_b \subseteq U$ and $\mathbb{X}^c_c \subseteq V$.

Consider the following diagram of spaces and maps:

$$\begin{tikzcd}[column sep=tiny]
 & & & & \mathrm{H}(\mathbb{X}^c_b) & & & & \\
 & \mathrm{H}(\mathbb{X}^b_a) & & \mathrm{H}(U) \arrow[ur] & & \mathrm{H}(V) \arrow[ul] & & \mathrm{H}(\mathbb{X}^d_c) & \\
\mathrm{H}(\mathbb{X}^a_a) \arrow[ur] & & \mathrm{H}(\mathbb{X}^b_b) \arrow[ul] \arrow[ur] & & \mathrm{H}(Y) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}^c_c) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}^d_d) \arrow[ul]
\end{tikzcd}$$

By the Restriction and Diamond Principles (since Y is H-taut) we have

$$\Big\langle$$  $$\Big\rangle = \Big\langle$$  $$\Big\rangle = \Big\langle$$  $$\Big\rangle \leq \mathrm{H}(Y) < \infty.$$

The result now follows from Lemma 3.9. □

3.5. **The Four Diagrams of Parametrized Homology.** Let $\mathbb{X} = (X, f)$ be an $\mathbb{R}$-space and let H be a homology functor with field coefficients. Quantities $\mu_{\mathbb{X}}^{\backslash\backslash}$, $\mu_{\mathbb{X}}^{\vee}$, $\mu_{\mathbb{X}}^{\wedge}$, and $\mu_{\mathbb{X}}^{//}$ capture the way topological features of $\mathbb{X}$ perish at endpoints. When they are r-measures, each defines a persistence diagram via the Equivalence Theorem. We denote these four decorated persistence diagrams by $\mathrm{Dgm}^{\backslash\backslash}(\mathbb{X})$, $\mathrm{Dgm}^{\vee}(\mathbb{X})$, $\mathrm{Dgm}^{\wedge}(\mathbb{X})$, and $\mathrm{Dgm}^{//}(\mathbb{X})$. These, collectively, comprise the *parametrized homology* of $\mathbb{X}$ with respect to the homology functor H.

Theorem 3.11. *We can define parametrized homology of $\mathbb{X} = (X, f)$ when:*

(i) X is a locally compact polyhedron, f is proper, and H *is Steenrod–Sitnikov homology.*

(iii) X is a smooth manifold and f is a proper Morse function.

(iv) X is a locally compact polyhedron and f is a proper piecewise-linear map.

(v) $X \subseteq \mathbb{R}^n \times \mathbb{R}$ is a closed definable set in some o-minimal structure and f is the projection onto the second factor.

Proof. Additivity follows from Proposition 3.7 and finiteness from Proposition 3.10. □

3.6. **Levelset Zigzag Persistence.** In some situations finite zigzag diagrams carry all the needed information. Let $\mathbb{X} = (X, f)$ be an $\mathbb{R}$-space constructed as follows. There is a finite set of real-valued indices $S = \{a_1, ..., a_n\}$ (listed in increasing order), called the *critical values* of $\mathbb{X}$. Then:

- For $1 \leq i \leq n$, V_i is a locally path-connected compact space;
- For $1 \leq i \leq n-1$, E_i is a locally path-connected compact space;
- For $1 \leq i \leq n-1$, $l_i \colon E_i \to V_i$ and $r_i \colon E_i \to V_{i+1}$ are continuous maps.

Let X be the quotient space obtained from the disjoint union of the spaces $V_i \times \{a_i\}$ and $E_i \times [a_i, a_{i+1}]$ by making the identifications $(l_i(x), a_i) \sim (x, a_i)$ and $(r_i(x), a_{i+1}) \sim (x, a_{i+1})$ for all i and all $x \in E_i$. Let $f \colon X \to \mathbb{R}$ be the projection onto the second factor. In this paper, we follow Carlsson et al. [14] in calling such $\mathbb{R}$-spaces $\mathbb{X}$ *Morse type* $\mathbb{R}$-spaces. Vin de Silva et al. [20] on the other hand, call them *constructible* $\mathbb{R}$*-spaces*. They include $\mathbb{X} = (X, f)$, where X is a compact manifold and f a Morse function, and X a compact polyhedron and f piecewise linear.

We can track the appearance and disappearance of topological features using *levelset zigzag persistence* construction [14]. Given $\mathbb{X} = (X, f)$ of Morse type, select a set of indices s_i which satisfy

$$-\infty < s_0 < a_1 \ldots < a_n < s_n < \infty,$$

and build a zigzag diagram that serves as a model for $\mathbb{X}$:

$$\begin{tikzcd}[column sep=small]
 & & \mathbb{X}_{s_0}^{s_1} & & \mathbb{X}_{s_1}^{s_2} & & \cdots & & \mathbb{X}_{s_{n-2}}^{s_{n-1}} & & \mathbb{X}_{s_{n-1}}^{s_n} & \\
\mathbb{X}_{\{s_0,\ldots,s_n\}} : & \mathbb{X}_{s_0}^{s_0} \arrow[ur] & & \mathbb{X}_{s_1}^{s_1} \arrow[ul] \arrow[ur] & & \mathbb{X}_{s_2}^{s_2} \arrow[ul] \arrow[ur] & & \mathbb{X}_{s_{n-2}}^{s_{n-2}} \arrow[ul] \arrow[ur] & & \mathbb{X}_{s_{n-1}}^{s_{n-1}} \arrow[ul] \arrow[ur] & & \mathbb{X}_{s_n}^{s_n}. \arrow[ul]
\end{tikzcd}$$

Apply homology functor H to obtain:

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{H}(\mathbb{X}_{s_0}^{s_1}) & & \mathrm{H}(\mathbb{X}_{s_1}^{s_2}) & & \cdots & & \mathrm{H}(\mathbb{X}_{s_{n-2}}^{s_{n-1}}) & & \mathrm{H}(\mathbb{X}_{s_{n-1}}^{s_n}) & \\
\mathrm{H}\mathbb{X}_{\{s_0,\ldots,s_n\}} : & \mathrm{H}(\mathbb{X}_{s_0}^{s_0}) \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_1}^{s_1}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_2}^{s_2}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_{n-2}}^{s_{n-2}}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_{n-1}}^{s_{n-1}}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_n}^{s_n}). \arrow[ul]
\end{tikzcd}$$

This quiver representation is decomposable by Gabriel's Theorem [25].

We translate between the notation of intervals that appear in the levelset zigzag persistence of $\mathbb{X}$ and critical values as follows:

$$\begin{aligned}
[\mathrm{H}(\mathbb{X}_{i-1}^{i}), \mathrm{H}(\mathbb{X}_{j-1}^{j})] &\text{ corresponds to } [a_i, a_j] \text{ for } 1 \le i \le j \le n,\\
[\mathrm{H}(\mathbb{X}_{i-1}^{i}), \mathrm{H}(\mathbb{X}_{j-1}^{j})] &\text{ corresponds to } [a_i, a_j) \text{ for } 1 \le i < j \le n+1,\\
[\mathrm{H}(\mathbb{X}_{i}^{i}), \mathrm{H}(\mathbb{X}_{j-1}^{j})] &\text{ corresponds to } (a_i, a_j] \text{ for } 1 \le i \le j \le n,\\
[\mathrm{H}(\mathbb{X}_{i}^{i}), \mathrm{H}(\mathbb{X}_{j-1}^{j-1})] &\text{ corresponds to } (a_i, a_j) \text{ for } 1 \le i < j \le n+1.
\end{aligned}$$

We interpret a_0 as $-\infty$ and a_{n+1} as ∞.

The collection of these pairs of critical values, taken with multiplicity and labelled by the interval type is called the levelset zigzag persistence diagram of $\mathbb{X}$ and denoted by $\mathrm{Dgm}^{ZZ}(\mathrm{H}\mathbb{X})$.

The four quantities defined in Section 3.1, $\mu^{\wedge}_{\mathrm{H}\mathbb{X}}$, $\mu^{\backslash\backslash}_{\mathrm{H}\mathbb{X}}$, $\mu^{//}_{\mathrm{H}\mathbb{X}}$ and $\mu^{\vee}_{\mathrm{H}\mathbb{X}}$, are measures when $\mathbb{X}$ is of Morse type. Additivity follows from (ii) of Proposition 3.7, while finiteness from the assumption that all slices (ie. preimages of intervals) and levelsets have finite dimensional homology groups. In fact, parametrized homology and levelset zigzag persistence of a Morse type $\mathbb{R}$-space carry the same information, as the following theorem demonstrates.

Theorem 3.12. *If $\mathbb{X}$ is an $\mathbb{R}$-space of Morse type with critical values*

$$a_1 < a_2 < \ldots < a_n,$$

then the levelset zigzag persistence diagram of $\mathbb{X}$, $\mathrm{Dgm}^{ZZ}(\mathrm{H}\mathbb{X})$, contains the same information as the four diagrams $\mathrm{Dgm}^{\backslash\backslash}(\mathbb{X})$, $\mathrm{Dgm}^{\vee}(\mathbb{X})$, $\mathrm{Dgm}^{\wedge}(\mathbb{X})$, and $\mathrm{Dgm}^{//}(\mathbb{X})$. To be more precise,

$$\begin{aligned}
(a_i, a_j) \in \mathrm{Dgm}^{\wedge}(\mathrm{H}\mathbb{X}) &\text{ if and only if } (a_i^{+}, a_j^{-}) \in \mathrm{Dgm}^{ZZ}(\mathrm{H}\mathbb{X})\\
[a_i, a_j) \in \mathrm{Dgm}^{\backslash\backslash}(\mathrm{H}\mathbb{X}) &\text{ if and only if } (a_i^{-}, a_j^{-}) \in \mathrm{Dgm}^{ZZ}(\mathrm{H}\mathbb{X})\\
(a_i, a_j] \in \mathrm{Dgm}^{//}(\mathrm{H}\mathbb{X}) &\text{ if and only if } (a_i^{+}, a_j^{+}) \in \mathrm{Dgm}^{ZZ}(\mathrm{H}\mathbb{X})\\
[a_i, a_j] \in \mathrm{Dgm}^{\vee}(\mathrm{H}\mathbb{X}) &\text{ if and only if } (a_i^{-}, a_j^{+}) \in \mathrm{Dgm}^{ZZ}(\mathrm{H}\mathbb{X}).
\end{aligned}$$

Diagrams $\mathrm{Dgm}^{\wedge}(\mathrm{H}\mathbb{X})$, $\mathrm{Dgm}^{\backslash\backslash}(\mathrm{H}\mathbb{X})$, $\mathrm{Dgm}^{//}(\mathrm{H}\mathbb{X})$ and $\mathrm{Dgm}^{\vee}(\mathrm{H}\mathbb{X})$ contain no decorated points with nonzero multiplicity other than those specified above.

Proof. First we prove that if $[a_i, a_j]$ with multiplicity m, $m \ge 1$, is contained in the levelset zigzag persistence diagram of $\mathbb{X}$, then $\mathrm{m}_{\mathrm{Dgm}^{\vee}(\mathbb{X})}(a_i^{-}, a_j^{+}) = m$.

We select a set of indices s_i which satisfy

$$-\infty < s_0 < a_1 < s_1 < a_2 < \ldots < s_{n-1} < a_n < s_n < \infty.$$

By definition $[a_i, a_j]$ appears in the levelset zigzag persistence diagram with multiplicity m if and only if

$$\langle [\mathrm{H}(\mathbb{X}_{s_{i-1}}^{s_i}), \mathrm{H}(\mathbb{X}_{s_{j-1}}^{s_j})] \mid \mathrm{H}\mathbb{X}_{\{s_0,\ldots,s_n\}} \rangle = m.$$

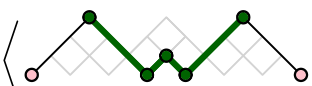
By the Diamond and the Restriction Principle

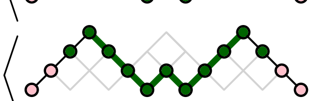
$$\langle [\mathrm{H}(\mathbb{X}_{s_{i-1}}^{s_i}), \mathrm{H}(\mathbb{X}_{s_{j-1}}^{s_j})] \mid \mathrm{H}\mathbb{X}_{\{s_0,\ldots,s_n\}} \rangle \quad = \quad \langle [\mathrm{H}(\mathbb{X}_{s_{i-1}}^{s_i}), \mathrm{H}(\mathbb{X}_{s_{j-1}}^{s_j})] \mid \mathrm{H}\mathbb{X}_{\{s_{i-1},s_i,s_{j-1},s_j\}} \rangle.$$

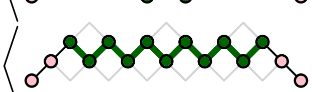
Choose $\epsilon < \frac{1}{2}\min\{a_i - s_{i-1}, s_j - a_j\}$. Observe the diagram below.

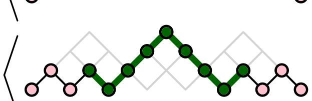
$$\begin{tikzcd}[column sep=tiny]
 & & & \mathrm{H}(\mathbb{X}_{s_{i-1}}^{s_i}) & & & & \mathrm{H}(\mathbb{X}_{a_i}^{a_j}) & & & & \mathrm{H}(\mathbb{X}_{s_{j-1}}^{s_j}) & & & \\
 & & \mathrm{H}(\mathbb{X}_{s_{i-1}}^{a_i}) \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_i-\epsilon}^{s_i}) \arrow[ul] & & \mathrm{H}(\mathbb{X}_{a_i}^{s_{j-1}}) \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_i}^{a_j}) \arrow[ul] & & \mathrm{H}(\mathbb{X}_{s_{j-1}}^{a_j+\epsilon}) \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_j}^{s_j}) \arrow[ul] & & \\
 & \mathrm{H}(\mathbb{X}_{s_{i-1}}^{a_i-\epsilon}) \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_i}^{a_i-\epsilon}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_i}^{s_i}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_i}^{s_{j-1}}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_{j-1}}^{a_j}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_j}^{a_j+\epsilon}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_j+\epsilon}^{s_j}) \arrow[ul] & \\
\mathrm{H}(\mathbb{X}_{s_{i-1}}^{s_{i-1}}) \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_i-\epsilon}^{a_i-\epsilon}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_i}^{a_i}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_i}^{s_i}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_{j-1}}^{s_{j-1}}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_j}^{a_j}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{a_j+\epsilon}^{a_j+\epsilon}) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_{s_j}^{s_j}) \arrow[ul]
\end{tikzcd}$$

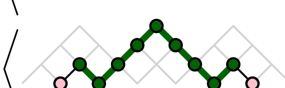
Using the Diamond Principle and the Restriction Principle we calculate:

$\langle [\mathrm{H}(\mathbb{X}_{s_{i-1}}^{s_i}), \mathrm{H}(\mathbb{X}_{s_{j-1}}^{s_j})] \mid \mathrm{H}\mathbb{X}_{\{s_{i-1},s_i,s_{j-1},s_j\}} \rangle$ $= \Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle$

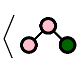
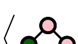
$$= \mu_{\mathrm{H}\mathbb{X}}^{\vee}([a_i - \epsilon, a_i] \times [a_j, a_j + \epsilon]).$$

In the second line we used the fact that $\mathbb{X}$ is of Morse type. This implies $\mathbb{X}_{s_{i-1}}^{s_{i-1}}$ is homotopy equivalent to $\mathbb{X}_{s_{i-1}}^{a_i-\epsilon}$, $\mathbb{X}_{s_{i-1}}^{a_i}$ to $\mathbb{X}_{s_{i-1}}^{s_i}$, $\mathbb{X}_{a_j+\epsilon}^{s_j}$ to $\mathbb{X}_{s_j}^{s_j}$ and $\mathbb{X}_{a_j}^{s_j}$ to $\mathbb{X}_{s_{j-1}}^{s_j}$ for all sufficiently small ϵ. Therefore

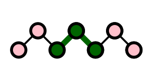
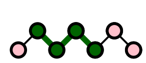
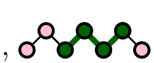
$$\mathrm{m}_{\mathrm{Dgm}^{\vee}(\mathbb{X})}(a_i^-, a_j^+) = \lim_{\epsilon \to 0} \mu_{\mathrm{H}\mathbb{X}}^{\vee}([a_i - \epsilon, a_i] \times [a_j, a_j + \epsilon]) = m.$$

We must now show that $\mathrm{Dgm}^{\vee}(\mathbb{X})$ contains only points of the type (a_i^-, a_j^+), where a_i and a_j are critical values of $\mathbb{X}$.

For any $p \in \mathbb{R}$, an $\epsilon > 0$ exists such that $\mathbb{X}_p^{p+\epsilon}$ and $\mathbb{X}_{p-\epsilon}^p$ strongly deformation retracts to $\mathbb{X}_p^p$. This means that $\mathrm{H}(\mathbb{X}_p^{p+\epsilon}) \cong \mathrm{H}(\mathbb{X}_p^p) \cong \mathrm{H}(\mathbb{X}_{p-\epsilon}^p)$, forcing

$\Big\langle$  $\mid \mathrm{H}\mathbb{X}_{\{p-\epsilon,p\}} \Big\rangle = \Big\langle$  $\mid \mathrm{H}\mathbb{X}_{\{p,p+\epsilon\}} \Big\rangle = 0.$

For ϵ small enough

, , 

all appear with 0 multiplicity for any p and q in the quiver decomposition of $\mathrm{H}\mathbb{X}_{\{p-\epsilon,p,q,q+\epsilon\}}$. This holds since by the restriction principle

$$0 \leq \left\langle \,\middle|\, \mathrm{H}\mathbb{X}_{\{p-\epsilon,p,q,q+\epsilon\}}) \right\rangle, \left\langle \,\middle|\, \mathrm{H}\mathbb{X}_{\{p-\epsilon,p,q,q+\epsilon\}}) \right\rangle \leq \left\langle \,\middle|\, \mathrm{H}\mathbb{X}_{\{p-\epsilon,p\}} \right\rangle = 0$$

and

$$0 \leq \left\langle \,\middle|\, \mathrm{H}\mathbb{X}_{\{p-\epsilon,p,q,q+\epsilon\}}) \right\rangle \leq \left\langle \,\middle|\, \mathrm{H}\mathbb{X}_{\{q,q+\epsilon\}} \right\rangle = 0.$$

So $\mathrm{Dgm}^{\vee}(\mathbb{X})$ contains exactly points that correspond to intervals of type $[a_i, a_j]$ in $\mathrm{Dgm}^{ZZ}(\mathrm{H}\mathbb{X})$. We prove the statement for other measures similarly. $\square$

3.7. **Sixteen behaviors.** Let $\mathbb{X}$ be an $\mathbb{R}$-space. Depending on the way a feature perishes and whether the corresponding interval is closed or open at endpoints, there are sixteen different cases that can occur (see Figure 4). For a Morse type $\mathbb{R}$-space $\mathbb{X} = (X, f)$, where X

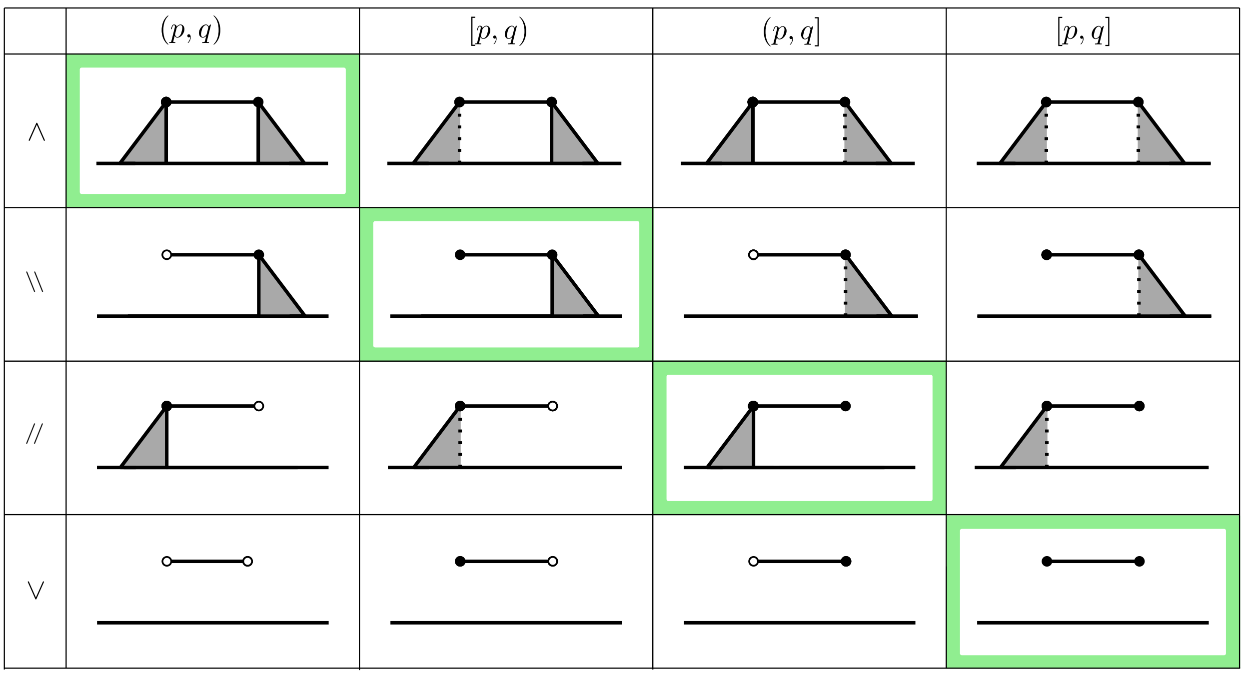


FIGURE 4. Different ways of dying at endpoints.

is compact, this number drops down to four (highlighted green in Figure 4) as demonstrated by Theorem 3.12. Something similar occurs when X is a locally compact polyhedron, f a proper continuous map and H the Steenrod–Sitnikov homology functor.

The following theorem, inspired by Frosini et al. [16], relies heavily on the continuity property of Čech homology [23]. For a wide variety of coefficient groups (infinitely divisible; finite exponent) [28] Čech homology coincides with Steenrod–Sitnikov homology. In particular, this is the case for some of the more common fields we may be interested in: $\mathbf{F}_p$, $\mathbf{Q}$, $\mathbf{R}$.

Theorem 3.13. *Let $\mathbb{X} = (X, f)$. We assume that X is a locally compact polyhedron, f is a proper continuous map, and* H *is the Steenrod–Sitnikov homology functor with coefficients in $\mathbf{F}_p$, $\mathbf{Q}$ or $\mathbf{R}$. Then:*

$$\begin{array}{ll}
\mathrm{Dgm}^{\wedge}(\mathrm{H}\mathbb{X}) & \textit{contains only points of type } \; = (p^+, q^-) = (p, q) \\
\mathrm{Dgm}^{\backslash\backslash}(\mathrm{H}\mathbb{X}) & \textit{contains only points of type } \; = (p^-, q^-) = [p, q) \\
\mathrm{Dgm}^{//}(\mathrm{H}\mathbb{X}) & \textit{contains only points of type } \; = (p^+, q^+) = (p, q] \\
\mathrm{Dgm}^{\vee}(\mathrm{H}\mathbb{X}) & \textit{contains only points of type } \; = (p^-, q^+) = [p, q]
\end{array}$$

In other words, the four possible decorations correspond exactly to the four ways in which a feature can perish at the ends of its interval.

Let $a < b < m < c < d$. We fix a piecewise-linear structure on X, and approximate $f\colon X \to \mathbb{R}$ with a piecewise-linear map $g\colon X \to \mathbb{R}$ for which $||g - f|| \leq \min\{\frac{c-m}{2}, \frac{m-b}{2}\}$. The preimage $Y = g^{-1}(m)$ is a finite simplicial complex. Let

$$V_q = g^{-1}((\infty, m]) \cap \mathbb{X}_q \quad \text{and} \quad U^q = g^{-1}([m, \infty)) \cap \mathbb{X}^q \;\text{ for } q \in \mathbb{R}.$$

In the proof of Theorem 3.13 we will make use of diagrams of this type:

$$\begin{tikzcd}[column sep=tiny]
 & & \mathrm{H}(V_a, \mathbb{X}_a^b) & & & & & & & & \mathrm{H}(U^d, \mathbb{X}_c^d) & \\
 & 0 \arrow[ur] & & \mathrm{H}(V_a, \mathbb{X}_a^a) \arrow[ul] & & & & & & \mathrm{H}(U^d, \mathbb{X}_d^d) \arrow[ur] & & 0 \arrow[ul] \\
\mathrm{H}\mathbb{X}^B_{\{a,b,c,d\}}\colon & & \mathrm{H}(\mathbb{X}_a^b, \mathbb{X}_a^a) \arrow[ul] \arrow[ur] & & \mathrm{H}(V_a) \arrow[ul] & & \mathrm{H}(\mathbb{X}_b^c) & & \mathrm{H}(U^d) \arrow[ur] & & \mathrm{H}(\mathbb{X}_c^d, \mathbb{X}_d^d) \arrow[ul] \arrow[ur] & \\
 & 0 \arrow[ur] & & \mathrm{H}(\mathbb{X}_a^b) \arrow[ul] \arrow[ur] & & \mathrm{H}(V_b) \arrow[ul] \arrow[ur] & & \mathrm{H}(U^c) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_c^d) \arrow[ul] \arrow[ur] & & 0 \arrow[ul] \\
 & & \mathrm{H}(\mathbb{X}_a^a) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_b^b) \arrow[ul] \arrow[ur] & & \mathrm{H}(Y) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_c^c) \arrow[ul] \arrow[ur] & & \mathrm{H}(\mathbb{X}_d^d) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Additionally, we will need the following lemma:

Lemma 3.14. *Let X be a compact subspace of a compact space Z, Y a finite simplicial complex contained in X and X_i a countable nested family of compact spaces such that $\cap_i X_i = X$. Let* H *be a Čech homology functor with coefficients in a field. In diagrams*

$$\mathrm{H}(X) \xrightarrow{j_i} \mathrm{H}(X_i) \quad \textit{and} \quad \mathrm{H}(Y) \xrightarrow{q_Y} \mathrm{H}(Z, X) \xrightarrow{q_i} \mathrm{H}(Z, X_i)$$

maps j_i, q_Y and q_i are induced by inclusions. The following equalities hold:

$$\cap_i \operatorname{Ker} j_i = 0 \quad \textit{and} \quad \operatorname{Ker} q_Y = \cap_i \operatorname{Ker} q_i \circ q_Y.$$

Proof. By continuity of Čech homology [23]

$$\varprojlim \mathrm{H}(Z, X_i) = \mathrm{H} \varprojlim (Z, X_i) = \mathrm{H}(Z, X).$$

The map

$$\mathrm{id}_{\mathrm{H}(Z,X)}\colon \varprojlim \mathrm{H}(Z, X_i) \longrightarrow \mathrm{H}(Z, X)$$

satisfies the compatibility conditions for inverse limits and by the universal property equals $\varprojlim q_i$. Similarly, $\varprojlim j_i = \mathrm{id}_{\mathrm{H}(X)}$.

Since the inverse limit functor preserves kernels,

$$\varprojlim \operatorname{Ker} j_i = \operatorname{Ker} \varprojlim j_i = \operatorname{Ker} \mathrm{id}_{\mathrm{H}(X)} = 0$$

and

$$\varprojlim \operatorname{Ker} q_i \circ q_Y = \operatorname{Ker} \varprojlim (q_i \circ q_Y) = \operatorname{Ker} \varprojlim q_i \circ \varprojlim q_Y = \operatorname{Ker} \mathrm{id}_{\mathrm{H}(Z,X)} \circ q_Y = \operatorname{Ker} q_Y.$$

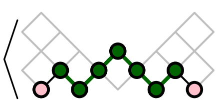
The statement follows since the inverse limit of a nested sequence of vector spaces is precisely their intersection. An identical argument proves the second statement. $\square$

Proof of Theorem 3.13. Let $(p, q) \in \mathbb{R}^2$ be such that $p < q < \infty$.

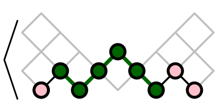
First we show that (p^+, q^*) appears with multiplicity 0 in $\mathrm{Dgm}^{\vee}(\mathbb{X})$ and $\mathrm{Dgm}^{\backslash\backslash}(\mathbb{X})$. It suffices to prove that

$$\lim_{\epsilon\to 0} \mu^{\vee}_{\mathbb{X}}([p, p+\epsilon] \times [c, d]) = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \mu^{\backslash\backslash}_{\mathbb{X}}([p, p+\epsilon] \times [c, d]) = 0.$$

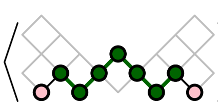
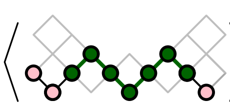
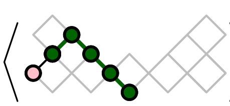
Let m and a descending sequence of positive numbers $\epsilon_1 \geq \epsilon_2 \geq \ldots \geq 0$ be such that $\lim\limits_{i\to\infty} \epsilon_i = 0$ and $p + 3\epsilon_1 < m < c - 3\epsilon_1$. Then

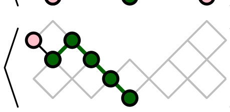
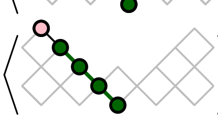
$$\mu^{\vee}_{\mathbb{X}}([p, p+\epsilon] \times [c, d]) = \Big\langle$$  $$\,|\, \mathrm{H}\mathbb{X}^B_{\{p,p+\epsilon,c,d\}} \Big\rangle$$

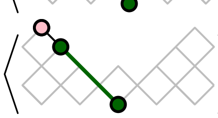
and

$$\mu^{\backslash\backslash}_{\mathbb{X}}([p, p+\epsilon] \times [c, d]) = \Big\langle$$  $$\,|\, \mathrm{H}\mathbb{X}^B_{\{p,p+\epsilon,c,d\}} \Big\rangle.$$

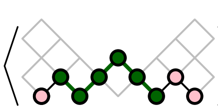
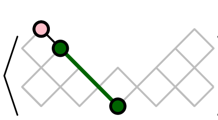
Using the Mayer–Vietoris and the restriction principles, we bound $\mu^{\vee}_{\mathbb{X}}([p, p+\epsilon] \times [c, d])$ and $\mu^{\backslash\backslash}_{\mathbb{X}}([p, p+\epsilon] \times [c, d])$:

$\Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle \leq \Big\langle$  $\Big\rangle$

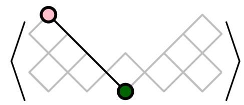
$\leq \Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle$

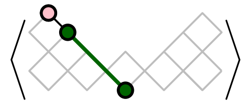
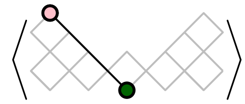
$= \Big\langle$  $\Big\rangle.$

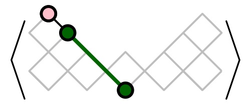
Similarly,

$\Big\langle$  $\Big\rangle \leq \Big\langle$  $\Big\rangle.$

By the restriction principle

$$\dim \operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p+\epsilon_i})) =$$ 

$$=$$  $+$ 

$$=$$  $+ \dim \operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p})).$

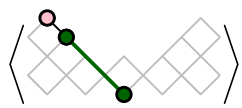
By Lemma 3.14

$$\cap_i \operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p+\epsilon_i})) = \operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p})).$$

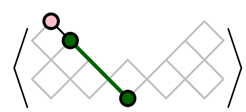
Since $\operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p+\epsilon_i}))$ and $\operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p}))$ are all finite dimensional,

$$\dim \operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p})) = \lim_{i \to \infty} \dim \operatorname{Ker} \mathrm{H}(Y \to (V_p, \mathbb{X}_p^{p+\epsilon_i})).$$

This implies that

$$\lim_{i\to\infty}$$  $= 0.$

For all i

$$0 \le \mu_{\mathbb{X}}^{\vee}([p, p+\epsilon_i] \times [c,d]), \mu_{\mathbb{X}}^{\backslash\backslash}([p, p+\epsilon_i] \times [c,d]) \le$$  .

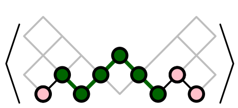
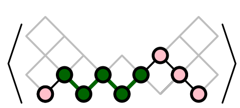
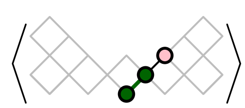
As we let $i \to \infty$, the desired statement follows.

By symmetry (p^*, q^-) appears with multiplicity 0 in $\mathrm{Dgm}^{\vee}(\mathbb{X})$ and $\mathrm{Dgm}^{//}(\mathbb{X})$.

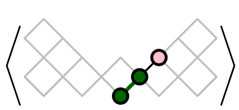
Next we prove that (p^*, q^+) appears with multiplicity 0 in $\mathrm{Dgm}^{\backslash\backslash}(\mathbb{X})$ and $\mathrm{Dgm}^{\wedge}(\mathbb{X})$, ie.

$$\lim_{\epsilon\to 0} \mu_{\mathbb{X}}^{\backslash\backslash}([a,b] \times [q, q+\epsilon]) = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \mu_{\mathbb{X}}^{\wedge}([a,b] \times [q, q+\epsilon]) = 0.$$

Let m and a descending sequence of positive numbers $\epsilon_1 \ge \epsilon_2 \ge \ldots \ge 0$ be such that $\lim\limits_{i\to\infty} \epsilon_i = 0$ and $b + 3\epsilon_1 < m < q - 3\epsilon_1$. Since all the diamonds are Mayer–Vietoris

 $=$  $\le$  .

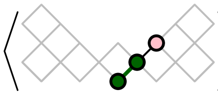
Note that

 $= \dim \left[\operatorname{Ker} \mathrm{H}(U^q \to U^{q+\epsilon_i}) \cap \operatorname{Im} \mathrm{H}(Y \to U^q). \right]$

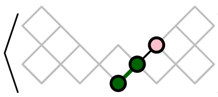
Vector spaces $\operatorname{Ker} \mathrm{H}(U^q \to U^{q+\epsilon_i}) \cap \operatorname{Im} \mathrm{H}(Y \to U^q)$ are finite dimensional subspaces of $\operatorname{Ker} \mathrm{H}(U^q \to U^{q+\epsilon_i})$ (Y is a finite simplicial complex and therefore has finitely generated homology groups). By Lemma 3.14 (it applies since Steenrod–Sitnikov and Čech homology coincide for a certain choice of coefficients)

$$\cap_i \operatorname{Ker} \mathrm{H}(U^q \to U^{q+\epsilon_i}) = 0.$$

Consequently,

$\lim_{i\to\infty} \Big\langle$  $\Big\rangle = \lim_{i\to\infty} \dim \operatorname{Ker} \mathrm{H}(U^q \to U^{q+\epsilon_i}) \cap \operatorname{Im} \mathrm{H}(Y \to U^q) = 0.$

Since

$0 \leq \mu_{\mathbb{X}}^{\backslash\backslash}([a,b] \times [q, q+\epsilon_i]) \leq \Big\langle$  $\Big\rangle,$

$\lim_{i\to\infty} \mu_{\mathbb{X}}^{\backslash\backslash}([a,b]\times[q,q+\epsilon_i]) = 0$ and consequently (p^-, q^*) appears with multiplicity 0 in the diagram determined by $\mu_{\mathbb{X}}^{\backslash\backslash}$. If we bound $\mu_{\mathbb{X}}^{\backslash\backslash}([a,b]\times[q,q+\epsilon])$ by the same term, we also get $\lim_{\epsilon\to 0} \mu_{\mathbb{X}}^{\wedge}([a,b]\times[q,q+\epsilon]) = 0$.

By symmetry (p^-, q^*) appears with multiplicity 0 in $\mathrm{Dgm}^{//}(\mathbb{X})$ and $\mathrm{Dgm}^{\wedge}(\mathbb{X})$. The statement follows.

□

Remark. The statement of Theorem 3.13 can be strengthened to include parametrized spaces (X, f), where:

- X is a Euclidean neighborhood retract and f is a proper continuous map (see [11]). This works because such an f can be approximated with a continuous g whose slices and levelsets are retracts of finite simplicial complexes and therefore have finitely generated homology groups.
- X is a compact ANR and f is a continuous function (see [10, 9, 8]). Any f can be approximated by a continuous map g whose slices and levelsets are compact ANR. Compact ANR's have finitely generated homology groups [30].

3.8. **Stability.** Given an $\mathbb{R}$-space $\mathbb{X} = (X, f)$ with a well-defined parametrized homology, what is the effect on the persistence diagrams of a small perturbation of the function? Will the resulting diagram be 'close' to the original? We can measure this in terms of the *bottleneck distance*, a standard and widely used metric on persistence diagrams [18].

The bottleneck distance compares undecorated diagrams. Let A, B be locally finite multisets defined in open sets $\mathcal{F}_A, \mathcal{F}_B$ in the extended plane $\overline{\mathbb{R}}^2$. Consider a partial bijection $\approx$ between A and B. The 'cost' of a partial bijection is defined

$$\operatorname{cost}(\approx) = \sup \begin{cases} \mathrm{d}^\infty((p,q),(r,s)) & \text{matched pairs } (p,q) \approx (r,s) \\ \mathrm{d}^\infty((p,q), \overline{\mathbb{R}}^2 - \mathcal{F}_B) & \text{if } (p,q) \in A \text{ is unmatched} \\ \mathrm{d}^\infty((r,s), \overline{\mathbb{R}}^2 - \mathcal{F}_A) & \text{if } (r,s) \in B \text{ is unmatched} \end{cases}$$

and the bottleneck distance is then

$$\mathrm{d_b}(A, B) = \inf \{\operatorname{cost}(\approx) \mid \approx \text{ is a partial bijection between } A \text{ and } B\}$$

One can show using a compactness argument that the infimum is attained [17]. In the definition we are using the l^∞-metric in the extended plane,

$$\mathrm{d}^\infty((p,q),(r,s)) = \max\{|p-r|, |q-s|\}$$

with $|(+\infty) - (+\infty)| = |(-\infty) - (-\infty)| = 0$. The distance to a subset is defined in the usual way. Note that the distance to $\overline{\mathbb{R}}^2 - \mathcal{H}$ is equal to the distance to the diagonal, that being the more familiar formulation.

We reach our stability theorem for parametrized homology (Theorem 3.17) by using a stability theorem from [17] for diagrams of r-measures. There is a natural way to compare two r-measures. For $R = [a,b] \times [c,d]$ define the δ-thickening $R^\delta = [a-\delta, b+\delta] \times [c-\delta, d+\delta]$. (For infinite rectangles, we use $-\infty - \delta = -\infty$ and $+\infty + \delta = +\infty$.) We say that two r-measures satisfy the *box inequalities with parameter* δ if

$$\mu(R) \leq \nu(R^\delta), \quad \nu(R) \leq \mu(R^\delta)$$

for all R. Either inequality is deemed to be vacuously satisfied if R^δ exceeds the finite support of the measure on the right-hand side.

It is natural to hope that two measures μ, ν which satisfy the box inequalities with parameter δ will determine diagrams with bottleneck distance bounded by δ. This is unfortunately not true, and in fact there is no universal bound on the bottleneck distance between the two diagrams. However, with stronger assumptions, namely the existence of a 1-parameter family interpolating between μ and ν, such a statement holds.

Theorem 3.15 (Stability for finite measures [17])**.** *Suppose* $(\mu_t \,|\, t \in [0,\delta])$ *is a 1-parameter family of finite r-measures on* $\mathcal{H}$. *Suppose for all* $s, t \in [0,\delta]$ *the box inequality*

$$\mu_s(R) \leq \mu_t(R^{|s-t|})$$

holds for all R. *Then there exists a* δ*-matching between* $\mathrm{Dgm_u}(\mu_0)$ *and* $\mathrm{Dgm_u}(\mu_\delta)$. □

We now apply this to the situation at hand.

Lemma 3.16 (Box lemma)**.** *Let* $\mathbb{X} = (X, f)$, $\mathbb{Y} = (X, g)$ *be* $\mathbb{R}$*-spaces with* H*-taut fibers on the same total space* X. *Write* $\mu^{\between} = \mu^{\between}_{\mathrm{H}\mathbb{X}}$ *and* $\nu^{\between} = \mu^{\between}_{\mathrm{H}\mathbb{Y}}$ *for* $\between = \wedge, \backslash\backslash, //, \vee$. *Then*

$$\mu^{\between}(R) \leq \nu^{\between}(R^\delta) \quad \textit{and} \quad \nu^{\between}(R) \leq \mu^{\between}(R^\delta)$$

for any $\delta > \|f - g\|$.

Proof. We only need to consider rectangles $R = [a,b] \times [c,d]$ whose δ-thickening is contained in $\mathcal{H}$. This implies, in addition to $a < b < c < d$, that $b + \delta < c - \delta$.

The proof requires four different kinds of interlevelset. When $p \leq q$ we have the familiar

$$\mathbb{X}_p^q = \{x \in X \mid p \leq f(x) \leq q\}, \qquad \mathbb{Y}_p^q = \{x \in X \mid p \leq g(x) \leq q\},$$

and when $p + \delta \leq q$ we define two new kinds,

$$\mathbb{U}_p^q = \{x \in X \mid p \leq f(x), g(x) \leq q\}, \quad \mathbb{V}_p^q = \{x \in X \mid p \leq g(x), f(x) \leq q\}.$$

In other words, $\mathbb{U}_p^q$ is the space cut out between $f^{-1}(p)$ on the left and $g^{-1}(q)$ on the right. The condition $p + \delta \leq q$ ensures that $\mathbb{U}_p^q$ and $\mathbb{V}_p^q$ separate $\mathbb{X}$ in the obvious way:

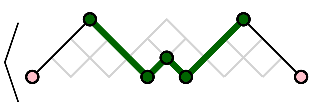
$$\begin{aligned} \mathbb{X} &= \mathbb{X}^p \cup \mathbb{U}_p^q \cup \mathbb{Y}_q \quad \text{with} \quad \mathbb{X}^p \cap \mathbb{U}_p^q = \mathbb{X}_p^p \quad \text{and} \quad \mathbb{U}_p^q \cap \mathbb{Y}_q = \mathbb{Y}_q^q \\ \mathbb{X} &= \mathbb{Y}^p \cup \mathbb{V}_p^q \cup \mathbb{X}_q \quad \text{with} \quad \mathbb{Y}^p \cap \mathbb{V}_p^q = \mathbb{Y}_p^p \quad \text{and} \quad \mathbb{V}_p^q \cap \mathbb{X}_q = \mathbb{X}_q^q. \end{aligned}$$

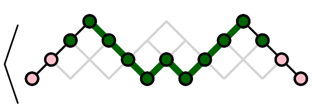
Consider the following Himalayan diagram:

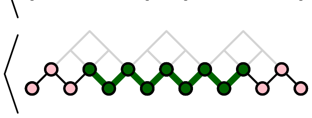
$$\begin{tikzcd}[column sep=tiny]
& & & \mathbb{Y}^{b+\delta}_{a-\delta} & & & & \mathbb{X}^{c}_{b} & & & & \mathbb{Y}^{d+\delta}_{c-\delta} & & & \\
& & \mathbb{V}^{b}_{a-\delta} \arrow[ur] & & \mathbb{U}^{b+\delta}_{a} \arrow[ul] & & \mathbb{U}^{c-\delta}_{b} \arrow[ur] & & \mathbb{V}^{c}_{b+\delta} \arrow[ul] & & \mathbb{V}^{d}_{c-\delta} \arrow[ur] & & \mathbb{U}^{d+\delta}_{c} \arrow[ul] & & \\
& \mathbb{V}^{a}_{a-\delta} \arrow[ur] & & \mathbb{X}^{b}_{a} \arrow[ul] \arrow[ur] & & \mathbb{U}^{b+\delta}_{b} \arrow[ul] \arrow[ur] & & \mathbb{Y}^{c-\delta}_{b+\delta} \arrow[ul] \arrow[ur] & & \mathbb{V}^{c}_{c-\delta} \arrow[ul] \arrow[ur] & & \mathbb{X}^{d}_{c} \arrow[ul] \arrow[ur] & & \mathbb{U}^{d+\delta}_{d} \arrow[ul] \arrow[dr] & \\
\mathbb{Y}^{a-\delta}_{a-\delta} \arrow[ur] & & \mathbb{X}^{a}_{a} \arrow[ul] \arrow[ur] & & \mathbb{X}^{b}_{b} \arrow[ul] \arrow[ur] & & \mathbb{Y}^{b+\delta}_{b+\delta} \arrow[ul] \arrow[ur] & & \mathbb{Y}^{c-\delta}_{c-\delta} \arrow[ul] \arrow[ur] & & \mathbb{X}^{c}_{c} \arrow[ul] \arrow[ur] & & \mathbb{X}^{d}_{d} \arrow[ul] \arrow[ur] & & \mathbb{Y}^{d+\delta}_{d+\delta}
\end{tikzcd}$$

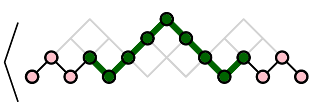
The nine diamonds of this diagram are Mayer–Vietoris. This is automatic for the top three diamonds. For the lower six diamonds we use the H-tautness of the fibers of $\mathbb{X}$ and $\mathbb{Y}$, and the fact that the space at the top of each diamond is a normal neighborhood of the fiber, since $\delta > \|f - g\|$.

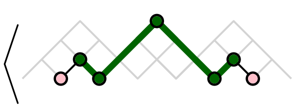
Applying H to the diagram, we calculate (for example):

$\nu^{\vee}(R^{\delta}) = \Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle$ + (eight other terms)

$= \Big\langle$  $\Big\rangle$ + (eight other terms)

$= \Big\langle$  $\Big\rangle$ + (eight other terms)

$= \Big\langle$  $\Big\rangle$ + (eight other terms)

$= \mu^{\vee}(R)$ + (eight other terms).

To explain the second line, note that there are nine different summand types which restrict to the summand type in the first line: three possible start points ($\mathbb{V}^{a}_{a-\delta}, \mathbb{V}^{b}_{a-\delta}, \mathbb{Y}^{b+\delta}_{a-\delta}$) times three possible end points ($\mathbb{Y}^{d+\delta}_{c-\delta}, \mathbb{U}^{d+\delta}_{c}, \mathbb{U}^{d+\delta}_{d}$). We are interested in only one of the nine terms.

Since the eight other terms are nonnegative, it follows that $\mu^{\vee}(R) \leq \nu^{\vee}(R^{\delta})$ for all relevant R. By symmetry, $\nu^{\vee}(R) \leq \mu^{\vee}(R^{\delta})$ also. The calculations for ⩙ $= \wedge, \backslash\backslash, //$ are similar. $\square$

Theorem 3.17 (Stability of parametrized homology)**.** *Let* $\mathbb{X} = (X, f)$ *and* $\mathbb{Y} = (X, g)$ *be* $\mathbb{R}$*-spaces with the same total space* X *that satisfy one of the following conditions:*

(i) X *is a locally compact polyhedron,* f *and* g *are proper, and* H *is Steenrod–Sitnikov homology.*

(iv) X *is a locally compact polyhedron,* f *and* g *are proper piecewise-linear maps.*

(v) $X \subseteq \mathbb{R}^n \times \mathbb{R}$ *is a closed definable set in some o-minimal structure and* f *is the projection onto the second factor.*

The associated r-measures for $\mathbb{X}, \mathbb{Y}$ are written with the letters μ, ν respectively. Then

$$\mathrm{d_b}(\mathrm{Dgm}_{\mathrm{u}}^{\bowtie}(\mathrm{H}\mathbb{X}), \mathrm{Dgm}_{\mathrm{u}}^{\bowtie}(\mathrm{H}\mathbb{Y})) \leq \|f - g\|$$

for each type $\bowtie = \wedge, \backslash\!\backslash, /\!/, \vee$.

Proof. For any $\delta > \|f - g\|$ we can define the interpolating family

$$f_t = (1 - (t/\delta))f + (t/\delta)g$$

for $t \in [0, \delta]$. Note that $f_0 = f$ and $f_\delta = g$. Since f is proper and $\|f - f_t\|$ is bounded for all $t \in [0, \delta]$, f_t are proper. So each (X, f_t) in situations (i) and (iv) determines an r-measure $\mu_t^{\bowtie}$. For any $s, t \in [0, \delta]$ we have $\|f_s - f_t\| < |s - t|$ and therefore

$$\mu_s^{\bowtie}(R) \leq \mu_t^{\bowtie}(R^{|s-t|})$$

by Lemma 3.16. Theorem 3.15 implies that there exists an δ-matching between

$$\mathrm{Dgm_u}(\mu_0^{\bowtie}) = \mathrm{Dgm_u}(\mu^{\bowtie}) = \mathrm{Dgm}_{\mathrm{u}}^{\bowtie}(\mathrm{H}\mathbb{X}) \quad \text{and} \quad \mathrm{Dgm_u}(\mu_{\|f-g\|}^{\bowtie}) = \mathrm{Dgm_u}(\nu^{\bowtie}) = \mathrm{Dgm}_{\mathrm{u}}^{\bowtie}(\mathrm{H}\mathbb{Y}).$$

Since this is true for all $\delta > \|f - g\|$ the result follows. $\square$

3.9. **Extended persistence.** Closely related to ours is the work on *extended persistence* by Cohen-Steiner, Edelsbrunner, and Harer [19]. Among other contributions, they construct four types of diagrams associated with an $\mathbb{R}$-space. These diagrams can describe the geometry and topology of a three-dimensional shape, a feature that finds applications in protein docking [2]. In this section we explain how their four diagrams correspond exactly with the four parametrized homology measures we have developed in this paper.

Given an $\mathbb{R}$-space $\mathbb{X} = (X, f)$ they examine a concatenation of two sequences of spaces: a filtration of the sublevelsets of f and a filtration of pairs of the space relative to the superlevelsets of f.

$$\mathbb{X}^{a_1} \to \mathbb{X}^{a_2} \to \ldots \to \mathbb{X}^{a_n} \to \mathbb{X} = (\mathbb{X}, \emptyset) \to (\mathbb{X}, \mathbb{X}_{a_n}) \to \ldots \to (\mathbb{X}, \mathbb{X}_{a_2}) \to (\mathbb{X}, \mathbb{X}_{a_1})$$

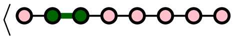
The indices $a_1, a_2, \ldots, a_n$ are taken to be the critical values of f; the underlying assumption of [19] being that we are in a Morse type situation.

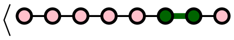
Within this sequence, four types of intervals are distinguished: those that are supported on the absolute (ordinary) half of the sequence, those supported on the relative half, and those supported over both halves, in the latter further distinguishing intervals where the superscript of the space associated to the left endpoint is lower or higher than the subscript in the relative part of the right endpoint.

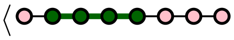
To translate their work into the language of measures, for real numbers $a < b < c < d$ we consider a sequence of spaces:

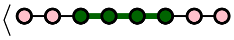
$$\mathbb{X}_{a,b,c,d}^{EP} : \mathbb{X}^a \to \mathbb{X}^b \to \mathbb{X}^c \to \mathbb{X}^d \to (\mathbb{X}, \mathbb{X}_d) \to (\mathbb{X}, \mathbb{X}_c) \to (\mathbb{X}, \mathbb{X}_b) \to (\mathbb{X}, \mathbb{X}_a).$$

We define four measures μ_i^{Ord}, μ_i^{Rel}, μ_i^{Rel}, μ_i^{Rel} for each non-negative index i as follows:

$\mu_i^{\mathrm{Ord}}([a,b]\times[c,d]) = \langle$  $\mid \mathrm{H}_i(\mathbb{X}^{EP}_{a,b,c,d})\rangle$

$\mu_i^{\mathrm{Rel}}([a,b]\times[c,d]) = \langle$  $\mid \mathrm{H}_i(\mathbb{X}^{EP}_{a,b,c,d})\rangle$

$\mu_i^{\mathrm{Ext}^+}([a,b]\times[c,d]) = \langle$  $\mid \mathrm{H}_i(\mathbb{X}^{EP}_{a,b,c,d})\rangle$

$\mu_i^{\mathrm{Ext}^-}([a,b]\times[c,d]) = \langle$  $\mid \mathrm{H}_i(\mathbb{X}^{EP}_{a,b,c,d})\rangle.$

In the case of a Morse type $\mathbb{R}$-space, we can retrieve the extended persistence intervals by restricting a, b, c, d to the critical values a_i of f. However, these four measures are defined without that assumption.

The main result of this section expresses the relationship between the extended persistence and the parametrized homology of the pair $\mathbb{X} = (X, f)$. Specifically, the four extended persistence measures are in one-to-one correspondence with the four parametrized homology measures.

Theorem 3.18. *Let* H *be a homology functor with field coefficients and* $\mathbb{X}$ *an* $\mathbb{R}$*-space with* H*-taut levelsets. Then:*

$$\begin{aligned} \mu_i^{\backslash\backslash} &= \mu_i^{\mathrm{Ord}} & \qquad \mu_i^{//} &= \mu_{i+1}^{\mathrm{Rel}} \\ \mu_i^{\vee} &= \mu_i^{\mathrm{Ext}^+} & \qquad \mu_i^{\wedge} &= \mu_{i+1}^{\mathrm{Ext}^-} \end{aligned}$$

Here we have abbreviated $\mu^{\mathbb{X}}_{\mathrm{H}_i\mathbb{X}}$ *to* $\mu_i^{\mathbb{X}}$ *for each type* $\mathbb{X} = \backslash\backslash, \vee, //, \wedge$.

Proof. We prove the third equality; the rest are proven similarly.

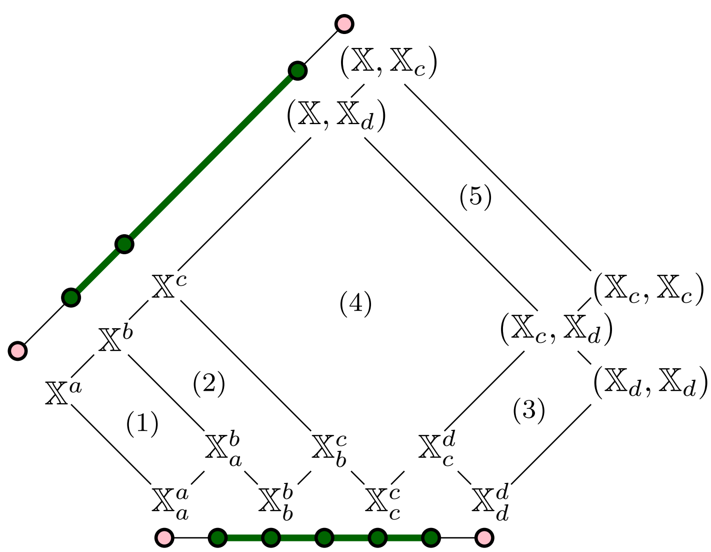
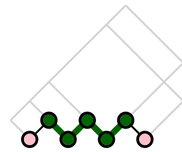
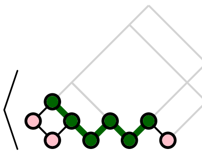
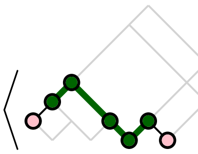
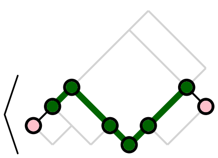
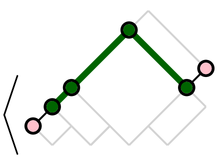
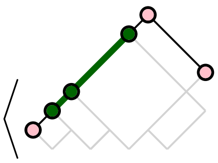
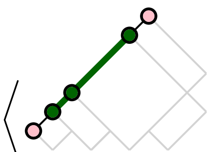


FIGURE 5. Diamonds involved in the proof $\mu_i^{\vee} = \mu_i^{\mathrm{Ext}^+}$.

Repeatedly applying the Diamond Principle to the spaces in Figure 5, we get

$$\mu_i^{\vee}([a,b]\times[c,d]) = \Big\langle$$  $$\Big\rangle = \Big\langle$$  $$\Big\rangle = \Big\langle$$  $$\Big\rangle$$

$$= \Big\langle$$  $$\Big\rangle = \Big\langle$$  $$\Big\rangle = \Big\langle$$  $$\Big\rangle$$

$$= \Big\langle$$  $$\Big\rangle = \mu_i^{\mathrm{Ext}^+}([a,b]\times[c,d])$$

for any rectangle $[a,b]\times[c,d]$. Thus the measures are equal. $\square$

4. Parametrized cohomology

Let $\mathbb{X} = (X, f)$ be a $\mathbb{R}$-space, and let H^* be a *cohomology* functor with coefficients in a field $\mathbf{k}$. We define four persistence measures, and therefore four persistence diagrams, just as we did with homology functors.

Remark. The formalism applies equally well to extraordinary cohomology functors (over $\mathbf{k}$).

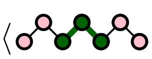
Here are the main steps. For any rectangle $R = [a,b]\times[c,d]$, the zigzag diagram of spaces

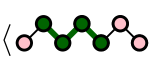
$$\mathbb{X}_{\{a,b,c,d\}}: \begin{tikzcd}[column sep=small] & \mathbb{X}_a^b & & \mathbb{X}_b^c & & \mathbb{X}_c^d & \\ \mathbb{X}_a^a \arrow[ur] & & \mathbb{X}_b^b \arrow[ul] \arrow[ur] & & \mathbb{X}_c^c \arrow[ul] \arrow[ur] & & \mathbb{X}_d^d \arrow[ul] \end{tikzcd}$$

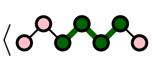
becomes a zigzag diagram of vector spaces

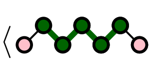
$$\mathrm{H}^*\mathbb{X}_{\{a,b,c,d\}}: \begin{tikzcd}[column sep=small] & \mathrm{H}^*(\mathbb{X}_a^b) \arrow[dl] \arrow[dr] & & \mathrm{H}^*(\mathbb{X}_b^c) \arrow[dl] \arrow[dr] & & \mathrm{H}^*(\mathbb{X}_c^d) \arrow[dl] \arrow[dr] & \\ \mathrm{H}^*(\mathbb{X}_a^a) & & \mathrm{H}^*(\mathbb{X}_b^b) & & \mathrm{H}^*(\mathbb{X}_c^c) & & \mathrm{H}^*(\mathbb{X}_d^d). \end{tikzcd}$$

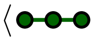
with the arrows reversed. Based on this diagram we define four measures

$$\mu^{\wedge}_{\mathrm{H}^*\mathbb{X}}(R) = \langle$$  $$\mid \mathrm{H}^*\mathbb{X}_{\{a,b,c,d\}}\rangle$$

$$\mu^{\setminus\setminus}_{\mathrm{H}^*\mathbb{X}}(R) = \langle$$  $$\mid \mathrm{H}^*\mathbb{X}_{\{a,b,c,d\}}\rangle$$

$$\mu^{//}_{\mathrm{H}^*\mathbb{X}}(R) = \langle$$  $$\mid \mathrm{H}^*\mathbb{X}_{\{a,b,c,d\}}\rangle$$

$$\mu^{\vee}_{\mathrm{H}^*\mathbb{X}}(R) = \langle$$  $$\mid \mathrm{H}^*\mathbb{X}_{\{a,b,c,d\}}\rangle$$

formally in the same way as before. The measures are additive if the fibers are H^*-taut (suitably defined), and finite if $\langle$  $\mid \mathrm{H}^*\mathbb{X}_{\{b,c\}}\rangle < \infty$. If both these conditions hold then four diagrams

$$\mathrm{Dgm}^{\wedge}(\mathrm{H}^*\mathbb{X}), \quad \mathrm{Dgm}^{\setminus\setminus}(\mathrm{H}^*\mathbb{X}), \quad \mathrm{Dgm}^{//}(\mathrm{H}^*\mathbb{X}), \quad \mathrm{Dgm}^{\vee}(\mathrm{H}^*\mathbb{X})$$

are defined. These diagrams constitute the parametrized cohomology of $\mathbb{X}$.

To a first approximation, there is no new information in parametrized cohomology.

Theorem 4.1. *If* H^* *is the cohomology functor dual to a homology functor* H*, then the four diagrams for* $\mathrm{H}^*\mathbb{X}$ *are equal to the respective four diagrams for* $\mathrm{H}\mathbb{X}$*.*

Proof. The universal coefficient theorem gives a natural isomorphism of functors $\mathrm{H}^*(-) \cong \mathrm{Hom}(\mathrm{H}(-), \mathbf{k})$. This implies that there is an isomorphism of zigzag modules

$$\mathrm{H}^*\mathbb{X}_{\{a,b,c,d\}} \cong \mathrm{Hom}(\mathrm{H}\mathbb{X}_{\{a,b,c,d\}}, \mathbf{k})$$

for every $a < b \leq c < d$. So it is sufficient to prove that any zigzag module $\mathbb{V}$ has the same interval-module multiplicities as its dual $\mathbb{V}^* = \mathrm{Hom}(\mathbb{V}, \mathbf{k})$. More precisely, Proposition 4.2 will show that the finite multiplicities agree. This is enough, because the construction of a diagram from its measure does not discriminate between different infinite cardinalities. $\square$

Proposition 4.2. *Let* $\mathbb{V}$ *be a zigzag module of length* n *and let* $\mathbb{V}^* = \mathrm{Hom}(\mathbb{V}, \mathbf{k})$ *be its dual. Then, for all* $1 \leq p \leq q \leq n$*, we have*

$$\langle [p,q] \mid \mathbb{V} \rangle = \langle [p,q] \mid \mathbb{V}^* \rangle,$$

with the understanding that all infinite cardinalities are regarded as equal.

Note that the shape of $\mathbb{V}^*$ is the shape of $\mathbb{V}$ with the arrows reversed, since $\mathrm{Hom}(-, \mathbf{k})$ is contravariant. We write $\mathbb{I}[p,q]$ to denote the interval module supported over $[p,q]$ that has the same arrow orientations as $\mathbb{V}$. The corresponding interval module with opposite arrow orientations can be identified with its dual $\mathbb{I}[p,q]^* \cong \mathrm{Hom}(\mathbb{I}[p,q], \mathbf{k})$.

Proof. An interval decomposition of $\mathbb{V}$ may be interpreted as an isomorphism

$$\mathbb{V} \cong \bigoplus_{p,q} V_{p,q} \otimes \mathbb{I}[p,q],$$

where the direct sum ranges over $0 \leq p \leq q \leq n$, and where the $V_{p,q}$ are vector spaces. The interval multiplicities of $\mathbb{V}$ are given by the formula $\langle \mathbb{I}[p,q] \mid \mathbb{V} \rangle = \dim(V_{p,q})$. We take the dual of both sides to obtain

$$\mathbb{V}^* \cong \bigoplus_{p,q} V^*_{p,q} \otimes \mathbb{I}[p,q]^*.$$

This depends on two standard facts: (i) the dual of a finite direct sum of vector spaces is naturally isomorphic to the direct sum of the duals of the vector spaces; and (ii) the dual of the tensor product of a vector space and a finite-dimensional vector space is naturally isomorphic to the tensor product of the duals of the two vector spaces. Thus

$$\langle [p,q] \mid \mathbb{V} \rangle = \langle \mathbb{I}[p,q] \mid \mathbb{V} \rangle = \dim(V_{p,q}) \stackrel{\text{fin}}{=} \dim(V^*_{p,q}) = \langle \mathbb{I}[p,q]^* \mid \mathbb{V}^* \rangle = \langle [p,q] \mid \mathbb{V}^* \rangle$$

where $x \stackrel{\text{fin}}{=} y$ means "x and y are equal or are both infinite". $\square$

In practice, one may choose to describe a given diagram as parametrized homology or cohomology according to whichever seems more natural in the given context. For example, here is a parametrized version of the classical Alexander duality theorem:

Theorem 4.3 (Parametrized Alexander Duality [26, 27]). *For $n \geq 2$, let $X \subset \mathbb{R}^n \times \mathbb{R}$, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let $p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the projection onto the second factor. We assume that (X, p) is proper, so that all levelsets $\mathbb{X}_a^a$ and slices $\mathbb{X}_a^b$ are compact. If parametrized Čech cohomology is defined for $\mathbb{X} = (X, p|_X)$, then it is also defined for $\mathbb{Y} = (Y, p|_Y)$. Additionally, for all $j = 0, \ldots, n-1$:*

$$\begin{aligned}
\mathrm{Dgm}^{\wedge}(\widetilde{\mathrm{H}}_{n-j-1}\mathbb{Y}) &= \mathrm{Dgm}^{\vee}(\check{\mathrm{H}}^j\mathbb{X}) \\
\mathrm{Dgm}^{\backslash\backslash}(\widetilde{\mathrm{H}}_{n-j-1}\mathbb{Y}) &= \mathrm{Dgm}^{//}(\check{\mathrm{H}}^j\mathbb{X}) \\
\mathrm{Dgm}^{//}(\widetilde{\mathrm{H}}_{n-j-1}\mathbb{Y}) &= \mathrm{Dgm}^{\backslash\backslash}(\check{\mathrm{H}}^j\mathbb{X}) \\
\mathrm{Dgm}^{\vee}(\widetilde{\mathrm{H}}_{n-j-1}\mathbb{Y}) &= \mathrm{Dgm}^{\wedge}(\check{\mathrm{H}}^j\mathbb{X})
\end{aligned}$$

For the proof, we refer to [26, 27]. Using this version of Alexander duality theorem, Henry Adams and Gunnar Carlsson [1] provide a criterion for the existence of an evasion path in a sensor network.

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